October 31, 1979

LIDS-FR-960

Research Supported By:

NASA Grant No. NSG 1312

ROBUSTNESS PROPERTIES OF DISCRETE TIME REGULATORS, LQG REGULATORS AND HYBRID SYSTEMS

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Final Report
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ABSTRACT

This report summarizes research accomplishments achieved under NASA Grant NSG-1312. Robustness properties of sample-data LQ regulators are derived which show that these regulators have fundamentally inferior uncertainty tolerances when compared to their continuous-time counterparts. New results are also presented in stability theory, multivariable frequency domain analysis, LQG robustness, and mathematical representations of hybrid systems.

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1. INTRODUCTION

Over the past several years, MIT's Laboratory for Information and Decision Systems (LIDS) has been conducting research for NASA on the properties of multivariable digital control systems. These types of systems are becoming increasingly important as small, powerful, flight-qualified digital computers take over the burden of control law implementation in various NASA vehicles and other control system applications. Examples include the shuttle orbiter, the HIMAT and F-8C DFBW aircraft, satellites such as ATS-6, various proposed large space systems, and many more.

The overall goal of the research program has been to evolve improved design methods for multivariable digital control laws. Research effort was concentrated initially on the primary available synthesis tool -- namely the sample-data (discrete-time) Linear-Quadratic (LQ) regulator problem [Athans, 1]. Various properties of this problem formulation were studied, and key features of its solution were investigated. In the latter category, the basic robustness properties of sample-data LQ solutions were studied under the specific NASA research grant NSG-1312. Research findings attributable to this grant are summarized in this report.

We will use the term "robustness" qualitatively to describe the ability of control system designs to maintain stability and performance in the face of plant uncertainties. The larger the level of uncertainties which can be tolerated, the more robust a design is considered to be. In real-life applications, robustness properties are among the most important features of control designs. This is true whether the designs are achieved with classical or modern synthesis methods, and whether they are implemented in analog or

digital fashion. In each case, the actual plant being controlled will invariably differ from the design model, thus necessitating a healthy measure of uncertainty tolerance. Further discussion of engineering motivations for robustness can be found in a tutorial paper by Stein, prepared in part under the NSG-1312 grant [2].

The report summarizes our robustness research in the form of seven short topical sections. Each section describes a major research area, briefly summarizes the principal findings and their significance, and cites published papers and/or appendices for further details. The major areas are the following:

Section 2 - Generalized Stability Theory

Section 3 - Robustness Guarantees for Sample Data Regulators

Section 4 - Frequency Domain Interpretations

Section 5 - System Specific Robustness Properties

Section 6 - Compensated Sample-Data Filters

Section 7 - LQG Robustness Properties

Section 8 - Hybrid System Descriptions

Three appendices are included to provide supporting details and derivations for topics where published manuscripts are not yet available.

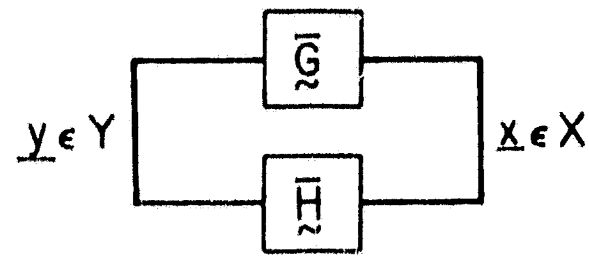
2. GENERALIZED STABILITY THEORY

Since stability is the foremost essential feature of feedback systems, its robustness properties with respect to plant uncertainties received primary research investigation. The major theoretical tools used for these investigations include the classical Nyquist and Lyapunov stability theories as well as a more abstract and general interpretation of stability due to Safonov. The latter was developed in part under the present grant.

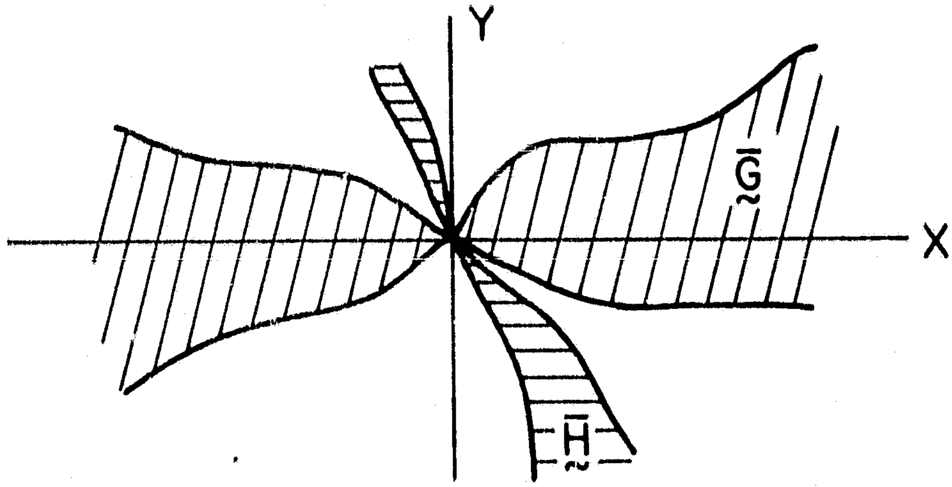
Stability as Topological Separation

In Safonov's interpretation, the stability property of a feedback system is viewed in an abstract yet elegantly simple way. The system is stable if its feedback and feedforward elements are appropriately separated in the function spaces on which they are defined. This notion is illustrated conceptually in Figure 1. Part A of this figure shows a standard feedback system with feedforward element $\bar{\underset{\sim}{G}}$ and feedback element $\bar{\underset{\sim}{H}}$. These elements are viewed quite abstractly as "relations" between their respective input functions and output function. This simply means that if we let $\mathcal{X}$ and $\mathcal{Y}$ denote the function spaces to which the points (functions) $\underline{x}$ and $\underline{y}$ belong, then $\bar{\underset{\sim}{G}}$ and $\bar{\underset{\sim}{H}}$ are subsets of the Cartesian product $\mathcal{X} \times \mathcal{Y}$. Part B of Figure 1 shows this interpretation schematically. $\mathcal{X}$ is represented as the real line (one axis), $\mathcal{Y}$ is another real line (the second axis), and $\mathcal{X} \times \mathcal{Y}$ is the plane. $\bar{\underset{\sim}{G}}$ and $\bar{\underset{\sim}{H}}$ are then two subsets of the plane.

It follows from this abstract view of $\bar{\underset{\sim}{G}}$ and $\bar{\underset{\sim}{H}}$, that all solutions of their feedback interconnection in Part A must be common points of the two subsets in Part B. Moreover, if it is known that these two subsets are separated such that (in the absence of disturbances and initial conditions)



Part A. Feedback System



Part B. Stable Separation

Figure 1. Safonov Stability Theory

they only have zero functions in common, as represented by the point $(0,0) \in \mathcal{X} \times \mathcal{Y}$, then the feedback system must be stable.

This simple statement of separation is the essence of Safonov's stability theory. Formally, of course, the theory requires much more elaboration and mathematical machinery. The function spaces $\mathcal{X}$ and $\mathcal{Y}$ must be extended inner product spaces, relations $\tilde{\underset{\sim}{G}}$ and $\bar{\underset{\sim}{H}}$ must account for disturbances and initial conditions through functional dependences of their own, and the notion of separation must be properly quantified. These formal developments are carried out in Safonov's Ph.D. thesis [3, Part 2] and in reference [4]. Only the major ideas needed to quantify separation are further discussed below.

Separating Functionals, Sectors, and the Multivariable Circle Criterion

The key idea which makes the above stability interpretation useful as a stability analysis tool is the concept of "separating functionals." These allow us to test whether the feedback system's elements indeed have only the origin in common. Very simply, a separating functional is any scalar valued function-of-functions $d(\underset{\sim}{x},\underset{\sim}{y})$ defined on the Cartesian product space $\mathcal{X} \times \mathcal{Y}$, whose sign separates this space into two regions. One region consists of all the points (pairs of functions) for which

$$d(\underset{\sim}{x},\underset{\sim}{y}) < 0 \quad , \tag{1}$$

and the other region consists of points

$$d(\underset{\sim}{x},\underset{\sim}{y}) > 0, \tag{2}$$

with $d(\underset{\sim}{x},\underset{\sim}{y}) = 0$ obviously forming the boundary. In terms of such separating functionals, a feedback system is stable if its two elements $\tilde{\underset{\sim}{G}}$ and $\bar{\underset{\sim}{H}}$ satisfy

(i) $d(x,y) \geq \eta(\underset{\sim}{x},\underset{\sim}{y}) \geq 0$

for all $(\underset{\sim}{x},\underset{\sim}{y})$ corresponding to $\underset{\sim}{\bar{G}}$, and (3)

(ii) $d(\underset{\sim}{x},\underset{\sim}{y}) < 0$

for all (x,y) corresponding to $\underset{\sim}{\bar{H}}$. (4)

Here $\eta(\underset{\sim}{x},\underset{\sim}{y})$ is a positive definite radially unbounded scalar functional which is imposed to assure a technical requirement that the subsets in Figure 1 grow "sufficiently far apart" as $\underset{\sim}{x}$ and/or $\underset{\sim}{y}$ get large.

Safonov has shown in [3] and [4] that the notion of stability as topological separation and established via separating functionals is quite general indeed. For example, the classical stability theory of Lyapunov can be derived by appropriate choices of $\underset{\sim}{\bar{G}}$, $\underset{\sim}{\bar{H}}$ and d[4]. Similarly, the SISO conic sector stability result of Zames [5], and hence its corollaries -- the Popov, circle, passivity, and small-gain stability criteria -- can also be derived from this perspective. In fact, Safonov has invented specific kinds of separating functionals which generalize Zame's concept of conic sectors and lead to a general multivariable version of the circle criterion.

Safonov's generalized conic sectors, simply called "sectors," are defined to be regions of the space $X \times Y$ which are based on the following specific separating functionals:

$$g(\underset{\sim}{x},\underset{\sim}{y}) = \underset{\sim}{F}(\underset{\sim}{x},\underset{\sim}{y}) \triangleq \langle \underset{\sim}{F}_{11}\underset{\sim}{y} + \underset{\sim}{F}_{12}\underset{\sim}{x},\ \underset{\sim}{F}_{21}\underset{\sim}{y} + \underset{\sim}{F}_{22}\underset{\sim}{x}\rangle \tag{5}$$

Here $\underset{\sim}{F}_{11}$ and $\underset{\sim}{F}_{21}$ are operators mapping Y into a third function space Z, $\underset{\sim}{F}_{12}$ and $\underset{\sim}{F}_{22}$ are operators mapping X into Z, and $\langle\cdot,\cdot\rangle$ denotes an inner product defined on Z. Then the "Sector of $\underset{\sim}{F}$" is the set of points $(\underset{\sim}{x},\underset{\sim}{y})$ for which

$$\underset{\sim}{F}(\underline{x},\underline{y}) \leq 0 \quad . \tag{6}$$

It follows from our previous interpretation of stability that the feedback system of Figure 1 will be stable if the subset of points corresponding to $\bar{\underset{\sim}{H}}$ is inside the sector of some functional $\underset{\sim}{F}$ while the subset of points corresponding to $\bar{\underset{\sim}{G}}$ is strictly outside that sector. Here the words "strictly outside" are used to imply the same kind in increasing separation with increasing $\underline{x}$ and/or $\underline{y}$ as was used in Condition (i) above.

In terms of this definition, the SISO conic sectors of Zames are simply the sectors of special functionals $\underset{\sim}{F}$ in the form

$$\left.\begin{aligned} \underset{\sim}{F}_{11} &= \underset{\sim}{F}_{21} = 1 \\ \underset{\sim}{F}_{12} &= -\ (c + r) \\ \underset{\sim}{F}_{22} &= -\ (c - r) \end{aligned}\right\}, \tag{7}$$

where c and r are scalars called the "center" and "radius" of the conic sector, respectively. These types of regions were used by Zames [5] to establish stability conditions which include the circle criterion [6] as a special case. The more general sectors were used by Safonov in [4] and [7] to prove a more general multivariable version of the circle criterion.

Without going into the derivations or formality in detail, the multivariable circle criterion was developed for a linear dynamic operator as the feedforward element, $\underset{\sim}{G}$, and a nonlinear dynamic operator as the feedback element, $\underset{\sim}{H}$. Suppose $\underset{\sim}{H}$ lies inside the generalized conic sector defined by

$$\underset{\sim}{F}(\underline{x},\underline{y}) = \langle \underline{y} - \underset{\sim}{C}\underline{x} - \underset{\sim}{R}\underline{x},\ \underline{y} - \underset{\sim}{C}\underline{x} + \underset{\sim}{R}\underline{x} \rangle \tag{8}$$

i.e., $\underset{\sim}{F}_{11} = \underset{\sim}{F}_{21} = I$, $\underset{\sim}{F}_{12} = -\underset{\sim}{C} - \underset{\sim}{R}$ and $\underset{\sim}{F}_{22} = -\underset{\sim}{C} + \underset{\sim}{R}$, where the multivariable center and radius, $\underset{\sim}{C}$ and $\underset{\sim}{R}$, are themselves linear dynamic operators, and where $\langle \cdot,\cdot \rangle$ denotes the standard inner product on extended L_2 function space (i.e., $\langle \underline{a},\underline{b} \rangle_\tau \triangleq \int_0^\tau a^T(t)b(t)dt,\ \forall\tau$). Then, according to the topological separation concept, the feedback system will be stable if all points corresponding to $\underset{\sim}{G}$ fall outside the sector of F, or equivalently, if for all points $\underline{x} = \underset{\sim}{G}\underline{y}$ we have

$$
\begin{aligned}
0 \le \underset{\sim}{F}(\underline{x},\underline{y}) &= \langle \underline{y} - \underset{\sim}{C}\underset{\sim}{G}\underline{y} - \underset{\sim}{R}\underset{\sim}{G}\underline{y},\ \underline{y} - \underset{\sim}{C}\underset{\sim}{G}\underline{y} + \underset{\sim}{R}\underset{\sim}{G}\underline{y} \rangle \\
&= ||(I - \underset{\sim}{C}\underset{\sim}{G})\underline{y}||^2 - ||\underset{\sim}{R}\underset{\sim}{G}\underline{y}||^2
\end{aligned} \tag{9}
$$

Using Parseval's theorem, the last expression can be transformed into the frequency domain to get the following sufficient stability criterion:

$$0 \le [I - CG(-j\omega)]^T[I - CG(j\omega)] - RG(-j\omega)^T RG(j\omega) \quad \text{for all } \omega \tag{10}$$

As a technical detail, it should be noted that the application of Parseval's theorem in the last step requires that the systems defined by transfer functions $G(I - CG)^{-1}$ and R must be themselves stable. This can be established by a separate Nyquist encirclement count or by explicit calculation of roots. For the sector radius, $\underset{\sim}{R}$, stability is usually imposed by assumption. Further details and other equivalent forms of (10) can be found in Safonov's thesis [3] and in the Safonov/Athans paper [7]. It is particularly interesting to note that (10) can be expressed in terms of singular values to get the stability robustness conditions of Doyle [13].

<u>Some Comments on Significance</u>

Aside from the obvious significance of the above results as a "global" theory encompassing various previous stability results as special cases, the

topological separation and sector concepts of Safonov have two specific features which make them invaluable for the research objectives of the current study. First, the abstract treatment of the two elements $\underset{\sim}{\overline{G}}$ and $\underset{\sim}{\overline{H}}$ makes no assumption about the underlying nature of these relations, e.g., whether they represent continuous or discrete devices. Hence, the stability results apply equally well to analog and digital control system analysis. Second, by their very nature, the results face up to the robustness equations. Stability is not assessed for specific system elements $\underset{\sim}{\overline{G}}$ and $\underset{\sim}{\overline{H}}$ but for a whole class of elements covered by the possible points within a sector. A feedback system which is stable for nominal elements $\underset{\sim}{\overline{H}}_0 \in \text{Sector } (\underset{\sim}{F})$, $\underset{\sim}{\overline{G}}_0^I \in \text{Sector } (\underset{\sim}{F})^{\perp}$ will remain stable for all perturbed elements $\underset{\sim}{\overline{H}}$ within that sector. Hence, the "size" of the sector, as measured, for example, by the magnitude of its radius, $\underset{\sim}{R}$, becomes an immediate indicator of the degree of robustness of the system. The utility of both of these features becomes evident below.

3. ROBUSTNESS GUARANTEES FOR SAMPLE DATA REGULATORS

For the research objectives of the NSG-1312 grant, we are specifically interested in discrete time or sample data representations of linear dynamic systems in the following form:

$$x_{k+1} = Ax_k + Bu_k \tag{11}$$

$$u_k = -Gx_k \tag{12}$$

Here x_k denotes the usual n-dimensional state vector corresponding to continuous system states sampled at discrete instants of time, u_k is an m-vector of controls which is constant over each sample interval, and A, B, G are matrices of appropriate dimension. We assume that the feedback gain G is obtained by solving a sample-data linear-quadratic regulator problem, and that it therefore satisfies the well known discrete time Riccati equations [1]:

$$G = (R + B^T KB)^{-1} B^T KA \tag{13}$$

$$K = (A - BG)^T K(A - BG) + Q + G^T RG \quad . \tag{14}$$

Under mild assumptions on A, B, Q and R, the resulting closed loop system is, of course, stable and can be made to exhibit desirable dynamic properties through appropriate manipulations of Q and R. The research question at hand is to quantify the extent to which these properties -- stability in particular -- will be maintained as the true system description in (11) - (12) deviates from the design model used to compute G in (13) - (14).

The stability theory summarized in Section 2 proves ideally suited to this research task. We note first that equations (11) - (12) provide very specific

forms for the general input-output relations $\tilde{G}$, $\tilde{H}$ considered earlier. The feedforward relation, $\tilde{G}$, for instance, can be taken to be the (nominally) algebraic map

$$\begin{aligned} \underline{x} &= \tilde{G}\underline{y} \\ &= (A - BG)\underline{y} \end{aligned} \tag{15}$$

and $\tilde{H}$ can be taken as a multivariable delay operator

$$\begin{aligned} \underline{y} &= \tilde{H}\underline{x} \\ &= \{y_{k+1} = x_k; \quad k = 0,1,\ldots, \text{ with } y_0 = 0\} \end{aligned} \tag{16}$$

Both are operators on the (extended) function space of n-dimensional sequences with inner product

$$\langle \underline{x}, \underline{y} \rangle_\tau \triangleq \sum_{k=1}^{\tau} x_k^T y_k \quad \forall \tau \tag{17}$$

Using (17) in Safonov's definition of sectors, it is then a simple matter to show that the points (pairs of sequences) corresponding to $\tilde{H}$ fall into sectors defined by

$$\tilde{F}(\underline{x}, \underline{y}) = \langle P^{1/2}\underline{y} - P^{1/2}\underline{x},\ P^{1/2}\underline{y} + P^{1/2}\underline{x} \rangle \tag{18}$$

for any positive definite symmetric matrix $P^{1/2}$. This is true because (18) evaluated at points satisfying (16) becomes

$$\begin{aligned} \tilde{F}(\underline{x}, \tilde{H}\underline{x}) &= \sum_{i=1}^{\tau} x_{i-1}^T P x_{i-1} - \sum_{i=1}^{\tau} x_i^T P x_i \\ &= -x_\tau^T P x_\tau \leq 0 \quad \forall \tau \quad . \end{aligned} \tag{19}$$

It then follows immediately that the feedback system (11) - (12) will be stable

whenever the sequences $(\underline{x},\underline{y})$ corresponding to (15) fall strictly outside the sector defined by (18). The entire complement of Sector $(\tilde{F})$ thus forms the permissible range of plant variations which do not compromise stability.

Robustness of State Feedback

The above observations lead to the following fundamental result on the inherent robustness of sample-data state feedback:

> Let G be an arbitrary state feedback gain matrix which stabilizes the nominal design model (i.e., A - BG is stable). Define P to be the solution of the following steady state Lyapunov equation
>
> $$P = (A - BG)^T P(A - BG) + S \quad , \qquad S = S^T \geq 0 \tag{20}$$
>
> Then the feedback system (11) - (12) remains stable for all perturbed systems $\tilde{A} - \tilde{B}G$, where $\tilde{A}$ and $\tilde{B}$ are perturbed matrices or even non-linear dynamic operators, provided that the points $(\underline{v},\underline{w})$ corresponding to
>
> $$\underline{w} = P^{1/2}(\tilde{A} - \tilde{B}G)\underline{v} \tag{21}$$
>
> fall strictly inside the conic sector
>
> $$\tilde{F}(\underline{v},\underline{w}) = <\underline{w} - P^{1/2}\underline{v},\ \underline{w} + P^{1/2}\underline{v}> \tag{22}$$
>
> More simply, the system remains stable whenever the perturbed system matrix $P^{1/2}(\tilde{A} - \tilde{B}G)$ lies within the bounds $\pm P^{1/2}$. This is illustrated schematically in Figure 2.

To prove this result, it is only necessary to show that the sector conditions (21) - (22) imply that all points $(\underline{x},\underline{y})$ corresponding to $\tilde{A} - \tilde{B}G$ are strictly outside some sector of the form (18). To do this, we note that in the feedback interconnection of $\tilde{\bar{G}}$ and $\tilde{\bar{H}}$, the input of $\tilde{\bar{G}}$ corresponds to $\underline{y} \equiv \underline{v}$ and the output of $\tilde{\bar{G}}$ corresponds to $\underline{x} \equiv P^{-1/2}\underline{w}$. Then

$$<\underline{w} - P^{1/2}\underline{v},\ \underline{w} + P^{1/2}\underline{v}> \leq 0$$

$$\Rightarrow \quad <P^{1/2}\underline{x} - P^{1/2}\underline{y},\ P^{1/2}\underline{x} + P^{1/2}\underline{y}> \leq 0$$

$$\Rightarrow \quad <P^{1/2}\underline{y} - P^{1/2}\underline{x},\ P^{1/2}\underline{y} + P^{1/2}x> \geq 0 \qquad \text{Q.E.D.}$$

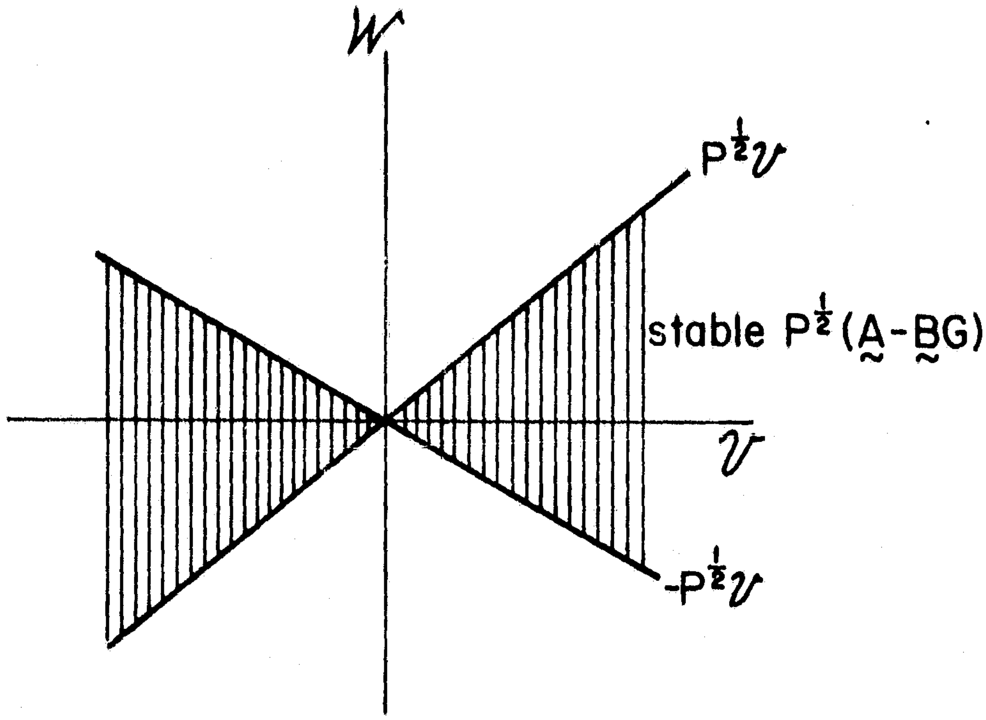


Figure 2. Sector of Stable State Feedback Systems

This is strictly a formal development, of course. A rigorous proof again requires the mathematics of extended function spaces, operators which are functionally dependent on disturbances, etc. Such proofs are developed in detail in Safonov's thesis [3, Part IV].

Robustness of Optimal Regulators

The above robustness result for sample-data state feedback applies directly to optimal regulators as well because these are known to satisfy the Riccati equation (14). Note that with G fixed, this equation is a Lyapunov equation like (20) with a specific choice of S. Hence, we can conclude directly that the LQ-regulator (11) - (12) remains stable for all perturbed operators $\underset{\sim}{A} - \underset{\sim}{B}G$ such that

$$\begin{aligned} &F(\underline{w},\underline{v}) = \langle \underline{w} - K^{1/2}\underline{v},\ \underline{w} + K^{1/2}\underline{v}\rangle \leq 0 \\ &\text{for all points } (\underline{w},\underline{v}) \text{ such that } \underline{w} = K^{1/2}(\underset{\sim}{A} - \underset{\sim}{B}G)\underline{v} \end{aligned} \tag{23}$$

Here again, the perturbed system $\underset{\sim}{A} - \underset{\sim}{B}G$ can consist of perturbed matrices A and B, or the matrices may themselves be nonlinear dynamic operators. This is evidently a very general robustness condition with various special applications.

Gain and Phase Margins

Two particularly meaningful applications concern the regulator's robustness with respect to specific perturbations such as gain changes, phase changes, or nonlinearities in the control channels. These manifest themselves as perturbed operators of the form

$$\underset{\sim}{A} - \underset{\sim}{B}G = A - B\underset{\sim}{N}G \tag{24}$$

where $\underset{\sim}{N}$ is a m x m nonlinear dynamic system nominally equal to identity. If we

assume that $\underset{\sim}{N}$ and the weighting matrix, R in (13) are both diagonal, then it is shown in [3] that the robustness condition (23) is satisfied whenever all diagonal elements of $\underset{\sim}{N}$ satisfy

$$f(\underline{w}_i,\underline{v}_i) = \langle \underline{w}_i - (c_i + r_i)\underline{v}_i,\ \underline{w}_i - (c_i + r_i)\underline{v}_i \rangle \leq 0$$

$$\text{for } c_i = \frac{1}{1 - a_i^2}$$

$$r_i = \frac{a_i}{1 - a_i^2}$$

$$a_i = \sqrt{R_{ii}/[R_{ii} + \lambda_{max}(B^T K B)]}$$

$$\text{and all } (\underline{w}_i,\underline{v}_i) \text{ such that } \underline{w}_i = \underset{\sim}{N}_{ii}\underline{v}_i \tag{25}$$

This condition requires that the points of each $\underset{\sim}{N}_{ii}$ must lie in a conic sector with center c_i and radius r_i, both defined by the weighting matrix elements R_{ii} and by the Riccati matrix K. If $\underset{\sim}{N}_{ii}$ is a pure algebraic gain or algebraic nonlinearity, for example, this requires that

$$c_i - r_i \leq \frac{N_{ii}(v_k)}{v_k} \leq c_i + r_i$$

$$\frac{1}{1 + a_i} \leq \frac{N_{ii}(v_k)}{v_k} \leq \frac{1}{1 - a_i} \tag{26}$$

for all $k = 1,2,\ldots$ Likewise, if $\underset{\sim}{N}_{ii}$ is a stable linear dynamic system, say $\underset{\sim}{L}_i$, then we must have

$$< (\underset{\sim}{L}_i - c_i)\underline{v}_i - r_i\underline{v}_i,\ (\underset{\sim}{L}_i - c_i)\underline{v}_i + r_i\underline{v}_i > \leq 0$$

$$\Rightarrow \quad ||(\underset{\sim}{L}_i - c_i)\underline{v}_i||^2 \leq ||r_i\underline{v}_i||^2$$

$$\Rightarrow \quad ||(L_i(e^{j\omega\Delta}) - c_i||^2 \leq r_i^2 \quad \forall\ \omega \tag{27}$$

where $L_i(z)$ is the z-transform of the operator $\underset{\sim}{L}_i$. This constraint confines $L_i(z)$, when evaluated at $z = e^{j\omega\Delta}$, to lie within a circle with center c_i and radius r_i. Given that $L_i(z)$ is nominally unity, it can therefore be perturbed in pure gain from $c_i - r_i = 1/(1+a_i)$ to $c_i + r_i = 1/(1-a_i)$ and is pure phase by

$$|\phi| \leq 2\ \sin^{-1}(a_i/2) \tag{28}$$

These then are the guaranteed gain and phase margins of the sample-data regulator. Note that they apply individually or in any combination to the m control channels.

Significance

The significance of the above margins can be be appreciated by noting that the scalars a_i in (25) - (27) are approximately unity. Their deviation from 1.0 is controlled by the quantity $\lambda_{max}(B^TKB)$ in (26) which is known to tend to zero as sample intervals Δ tend to zero ($B \to 0$ while $K \to$ const). Hence the a_i's approach unity from below and

Gain Margins $\to$ 1/2 to $+\infty$

Phase Margins $\to$ ±60 deg.

These limits are precisely the stability margins enjoyed by the continuous time linear-quadratic regulator [8]. The sample-data regulator, however, achieves these margins only asymptotically as sample rates get large. For all finite rates, it has fundamentally poorer margins.

A second important distinction between sample-data and continuous-time margins is that the latter are independent of plant and cost matrices. They are a consequence of optimality alone. In the sample data case, the parameters a_i depend on plant data (through R and $B^T KB$) and hence the margins are no longer global plant-independent guarantees.

Similar results as these apply to sample-data Kalman filters and to nonlinear systems linearized about various operating points $(\underline{x},\underline{y})$ as well. These additional results of the NSG-1312 grant are fully developed in Reference [3].

4. FREQUENCY DOMAIN INTERPRETATIONS

Both the continuous time margins in [8] and the sample data margins above were developed with relatively sophisticated mathematical machinery. This tends to make the results less accessible to practicing engineers than desirable. To overcome this problem, we have attempted under NSG-1312 to develop simple frequency domain explanations. These have proven quite useful in communicating the results and are briefly summarized below.

The Continuous-Time Case

The robustness properties of LQ-regulators can be viewed as multivariable generalizations of single-input frequency domain results dating back to Kalman. For the single-input case, Kalman proved that the return difference $T(s) \triangleq 1 + g^T(sI - A)^{-1}b$ of an optimal controller satisfies [9]

$$|T(j\omega)|^2 \geq 1 \tag{29}$$

at all frequencies, ω. Hence, the loop transfer function $G_0(s) \triangleq g^T(sI - A)^{-1}b$ lies outside of a unit circle centered at $(-1,j0)$ in the complex plane. This is illustrated in Figure 3 below:

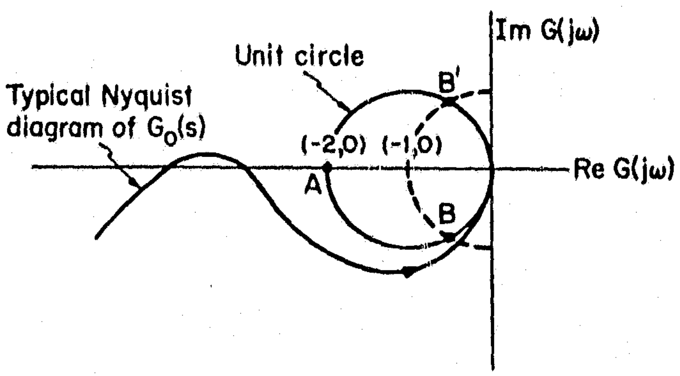


Figure 3. Optimal Nyquist Diagram

The (-6 to + ∞ db)-gain and $\pm$60 degree phase tolerances then follow from the Nyquist Stability Theorem applied to this geometry. Recall that the nominal system is stable. Hence, its encirclement count of the (-1,0) point is correct and will remain correct for all perturbations $G = G_0 + \varepsilon\delta G$ which do not cause the new Nyquist diagram to pass through (-1,0) for some $0 \leq \varepsilon < \infty$. If we consider perturbations which are pure gain changes only, for example, then $G = \varepsilon G_0$ and it is clear that the system remains stable for all ε, except when G_0 falls on the real axis for some ω, i.e., $G_0 = -\alpha + j0$. In that case, the tolerable ε range is $\frac{1}{\alpha} \leq \varepsilon < \infty$. Since α is guaranteed to be greater than or equal to 2.0 (Point A), the gain margin result follows. Similarly, if we consider pure phase changes such that $G = e^{j\varepsilon}G_0$ it follows that the system remains stable for all ε unless $|G_0| = 1$. In that case, the Points B and B' are the worst locations for G_0 and the $\pm$60 deg. phase margin property follows.

In terms of the multivariable generalization, it can be shown that the matrix version of the optimal return difference also satisfies an inequality, namely

$$[I + G'(-j\omega)]^T[I + G'(j\omega)] \geq I \tag{30}$$

This inequality implies that the loop transfer matrix $G'(s) \triangleq R^{1/2}G(s)R^{-1/2}$ lies outside of a unit ball centered at $(-I,j\underline{0})$ in the m-dimensional space of complex numbers. The (-6 to + ∞ db)-gain or $\pm$60 degree phase tolerances for each control channel then follow from the geometry of this ball. We again invoke Nyquist's Stability Theorem which now requires that the function $\det[I + G'(j\omega)]$ encircle the origin a requisite number of times. This number is correct for the nominal system and will remain correct as long as $I + G'(j\omega)$ remains nonsingular. This is assured as long as

$$(I + G')v \neq 0 \tag{31}$$

for all unit vectors v. However, from (30), we have that

$$|(I + G')v| \geq |v| \tag{32}$$

This means that G'v lies outside of a ball centered at -v with radius $|v|$. Projected onto any plane, the geometry of this ball looks just like Figure 3 and hence the allowable perturbations in G' follow from the same geometric arguments.

The Discrete Time Case

The analogous property to (29) for optimal sample data systems is

$$|1 + G(z)|^2 \geq \frac{r}{r+b^TKb}\ ; \quad s = e^{j\omega\Delta}, \tag{33}$$

where r is the (scalar) control weight, K is the Riccati matrix and Δ is sample time.

This condition implies that G(z), like G(s), lies outside of a circle centered at (-1, j0), but with radius $\mu \triangleq [r/(r+b^TKb)]^{1/2}$ less than unity. Hence, from the Nyquist Stability Theorem and the geometry of this smaller circle, it is clear that gain increases by factors greater than $1/(1-\mu)$, gain decreases by factors greater than $1/(1+\mu)$, or phase changes less than ±60 degrees could produce instability. Moreover, the radius parameter μ, and hence the margins, are plant-specific because they depend on K and b.

This same argument carries over to multivariable problems where the return difference can be shown to satisfy

$$[I + G'(\tfrac{1}{z})]^T [I + G'(z)] \geq \underline{R}^{-1/2} R \, \underline{R}^{-1/2} \quad , \qquad z = e^{j\omega T} \tag{34}$$

with $\underline{R} = R + B^T K B$

and $G'(z) = \underline{R}^{1/2} G(z) \, \underline{R}^{-1/2}$.

Here the loop transfer function $G'(z)$ is seen to lie outside of an "ellipsoidal ball" with minimum radius

$$\mu = [\lambda_{min}(\underline{R}^{-1/2} R \, \underline{R}^{-1/2})]^{1/2}$$

less than unity. As above, the margin properties follow from the geometry of this ball.

As in Section 3, the radius of the ball above is seen to approach unity as $B^T K B$ approaches zero. Hence, the continuous time margins and plant independent robustness guarantees are recovered asymptotically as sample rates tend to infinity.

5. SYSTEM SPECIFIC ROBUSTNESS PROPERTIES

As we observed above, the robustness properties of continuous time LQ-regulators are quite profound theoretically. They hold with no mention of the actual plant being controlled or its performance index. The margins are a consequence of optimality alone! All that is needed are the usual existence and uniqueness assumptions for LQ controls. Moreover, we have shown by counter examples that the margins are the broadest which can be achieved without further reference to particular system characteristics [10].

These observations do not mean, of course, that it is useless to look for broader plant-specific tolerance bounds. In many design problems, for example, it may well be important to increase the 50% gain reduction tolerance (-6 db) all the way to 100% (i.e. open loop) in order to achieve system reliability. Results which indicate that this is possible for specific problems have been derived by Wong, Athans and Stein [10] in part under the NSG-1312 grant. A particular result from [10] is that tolerable gain reductions can be bounded by

$$\Lambda > 1/2[I - R^{1/2}G^TQ^{-1}GR^{1/2}]^{-1} \quad (Q \text{ invertible}) \tag{35}$$

where $\Lambda = \mathrm{diag}\ (\gamma_1,\ldots,\gamma_m)$ is a diagonal pure gain perturbation in the control channels.

In specific examples, these lower bounds have been shown to be equal to the system's linear critical gain, which means that they achieve the broadest tolerance region possible for the example. Note that the bounds also provide, for the first time, an explicit relationship between gain margins and quadratic weights.

Analogous plant-specific robustness properties for discrete-time systems

were developed wholly under NSG-1312 and are documented in detail in a draft manuscript included Appendix A.

6. COMPENSATED SAMPLE-DATA FILTERS

We remarked in Section 3 that the robustness results achieved via Safonov stability theory apply to Kalman filter designs as well as to regulators. This connection is explored fully in [3]. Under NSG-1312 we also explored ways to enhance the resulting inherent filter robustness by dynamically compensating the filters so as to remove estimation biases. The details of this work are reported in a paper by Lee and Athans [11].

The basic premise of this paper is that the residual process of a discrete-time filter will exhibit low frequency biases whenever modeling errors and slowly varying inputs are present simultaneously. These biases can be modeled approximately as random walk processes. They can be observed by monitoring the residuals, and hence, they can be estimated by an auxiliary filter which uses the residuals as its "measurement" sequence. When the auxiliary and original filters are combined, they generate a 2n + m dimensional composite system which is effectively immune to unmodeled low frequency error mechanisms. Derivations and examples of this compensation procedure are given in [11].

7. LQG ROBUSTNESS PROPERTIES

As discussed in Sections 3 and 4, our robustness results for sample-data LQ regulators show that the very act of sampling seems to impose a loss of uncertainty tolerance (less gain margin, less phase margin, etc.) when compared with continuous time LQ-regulators. This "loss of robustness" in discrete regulators is also exhibited by continuous time regulators with state estimation. Hence, the possibility that there may exist common underlying reasons or at least useful interrelations between these two phenomena motivated further studies of the LQG continuous time case.

Results of these further studies are described in detail in a draft manuscript included in Appendix B. Highlights of these results are briefly reviewed here. First, the most basic discovery is that LQG-regulators have no guaranteed uncertainty tolerances whatsoever. This was established directly by a small design example due to Doyle [12] which produces a technically legitimate LQG-regulator with arbitrarily small tolerance for gain uncertainty (gain perturbation of $\pm\ \varepsilon$, with ε arbitrarily small, cause instability). The main significance of this example is that it shows LQG robustness to be a design-specific property. For the research effort, it meant that instead of looking for global guarantees, we should seek out generic design situations in which tolerances are likely to be good or poor. For the latter, we should devise adjustment procedures to improve robustness. The following results along these lines have been developed.

1. Margin recovery with "adapted Kalman filters"

   If the Kalman filter in an LQG-implementation receives the correct control signal (e.g., as altered by gain uncertainties) the LQG controller has gain margins equivalent to the full

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state case. This result is stated and proven as Proposition 1 in Appendix B. It is also proven in a more abstract setting in [3].

2. Asymptotic Margin Recovery I

Full-state gain margins can be recovered asymptotically as the following ratio tends to infinity:

$$\frac{\min\limits_{x^T x=1} -x^T (A+H_0 C^T) x}{\max\limits_{x^T x=1} -x^T (A+BG_0^T) x} \tag{36}$$

Here, A,B,C are the system dynamics, input and output matrices, and G_0^T and H_0 are the controller and filter gain matrices respectively.

3. Asymptotic Margin Recover II

Full-state gain margins can be recovered asymptotically if the process noise covariance, Φ, in the Kalman filter design tends to infinity in the following special manner:

$$\Phi \rightarrow \phi_2 BB^T, \quad \phi_2 \rightarrow \infty \tag{37}$$

Here ϕ_2 denotes a scalar.

The two asymptotic recovery results are stated and proven as Propositions 2 and 4 in the appendix. They serve the important function of providing ways to adjust LQG design parameters in design situations where nominal model-motivated parameters produce excessively sensitive controllers.

4. General Gain Margin Bounds

LQG systems are stable for the following range of control gain variations:

$$G^T = G_0^T \tag{38}$$

$$\text{with } \lambda_- \; I < \Lambda < \lambda_+ \; I$$

$$\lambda_+ \triangleq 1 + 1/\sqrt{\omega_0}$$

$$\lambda_- \triangleq \max[1-1/x_0, \; 1/\lambda_+] \tag{39}$$

Here Λ is a diagonal matrix, $\omega_0 > 0$ is a scalar which can be

made small by proper selection of design parameters, $x_0 > 0$ is a scalar which can be brought close to unity. Hence, this result also provides a systematic way to improve gain margins of an LQG-design. It is stated and proven as Proposition 3.

More detailed statements, proofs, and discussions of these results are provided in Appendix B. In addition, further research directions are also outlined there which are worthy of continued research effort.

8. HYBRID SYSTEM DESCRIPTIONS

We have now seen that sample data LQ-regulators are fundamentally inferior to their continuous-time counterparts in the sense that their robustness properties are not as good. They share this inferiority to some extent with continuous LQG regulators, but the latter can recover their robustness losses at least asymptotically by appropriate filter redesign. The only way that sample-data regulators can recover these losses is apparently to increase the sample rate arbitrarily.

Motivated by these apparent limitations of the existing sample data LQ synthesis methodology, the research effort under NSG-1312 was re-directed toward more fundamental issues of digitally-implemented control systems. The first task of the redirected effort was to find a mathematical representation which properly captures both the continuous-time (analog) and the discrete-time (digital) processes which occur side by side in a digital control system. Such a representation was developed in what we call the "hybrid operator model" of the control process. This model provides an analog input-output view of the control process which explicitly includes sampling operations, digital calculations, hold operations, and continuous plant evolutions. The structure of this operator is summarized briefly below and in more detail in Appendix C. The latter is a draft manuscript of A. Kostovetsky's Master's thesis prepared under the research grant.

An immediate application of the hybrid operator is to explain the common use of prefilters in practical digital control systems. Simple norm calculations in Appendix C show that the hybrid operator will have unbounded gain (in an appropriate function space sense) as the sampling process tends toward the

ideal impulsive sampling normally assumed in sample-data theory. Physically, this means that it provides arbitrary amplification for certain inputs (e.g., noise). Non-impulsive sampling, as obtained with pre-filters, bounds this amplification.

The second task of the redirected research made use of the hybrid operator model to answer the following very basic approximation question: How well can digitally-implemented control laws mimic analog ones? More specifically, if samplers, holds and digital algorithms are all selected to best approximate a given linear, time-invariant analog system, how good can the approximation be? The answer to this question is elegantly simple and profound. The digitally-implemented system can exactly duplicate the impulse response matrix, $G(t-\theta)$, of the analog system at all points in the t, θ-plane except on a strip of width τ (sample time) along the main diagonal ($t=\theta$). Inside this strip, the hybrid system's impulse response must be zero on various triangular segments. We have accordingly named this region of approximation the "triangle strip." Details of the optimal sampling, hold, and digital function for this approximation are again summarized below and derived in detail in Appendix C.

The significance of the above approximation result is that it provides a simple and clear picture of the basic limitations inherent in digitally-implemented controls. Such controls are fundamentally inferior to their analog counterparts because they cannot utilize all the input data in the triangle strip. This limits bandwidth, restricts performance, and precludes robustness guarantees such as those enjoyed by the continuous-time LQ regulator. The precise quantitative way in which these limitations manifest themselves, however, is not yet understood and provides basic motivation for continued research efforts.

Hybrid Systems

We will consider digital control systems which can be represented by the block diagram of Figure 4. The three main functions associated with the controller block in this diagram are:

1. The sampling operation which converts M-dimensional analog inputs $u(t)$ on the interval $(\ell-1)\tau \leq t \leq \ell\tau$ into N-dimensional discrete samples ξ_ℓ, $\ell=1,2,\ldots,$
2. the digital algorithm which converts the N-dimensional sequences ξ_ℓ into L-dimensional sequences η_k, $k=0,1,\ldots,$ and
3. the hold operation which converts the L-dimensional sequences η_k into k-dimensional analog functions $v(t)$ on the interval $k\tau \leq t \leq (k+1)\tau$.

The system's sample time will be designated by the symbol τ. These three functions will be assumed to have the forms

$$\xi_\ell = \int_{(\ell-1)}^{\ell\tau} f_\ell(\theta)u(\theta)d\theta = \int_{(\ell-1)}^{\ell\tau} f_0(\theta-\ell\tau)u(\theta)d\theta \tag{40}$$

$$\eta_k = \sum_{\ell=1}^{k} D_{k\ell}\xi_\ell \tag{41}$$

$$v(t) = g_k(t)\eta_k = g_0(t-k\tau)\eta_k \tag{42}$$

The first of these equations is a simple analog convolution operation with weighting function (impulse response) $f_0(\lambda)$. This could be the weighting function of an analog prefilter, an approximate impulsive sample, or various other vector valued input averaging operations. Some examples are given in Appendix C. The second equation is a standard digital convolution with coefficients $D_{k\ell}$. The third is a generalized output hold operation with weighting function $g_0(\lambda)$. This could be a simple constant to represent the common

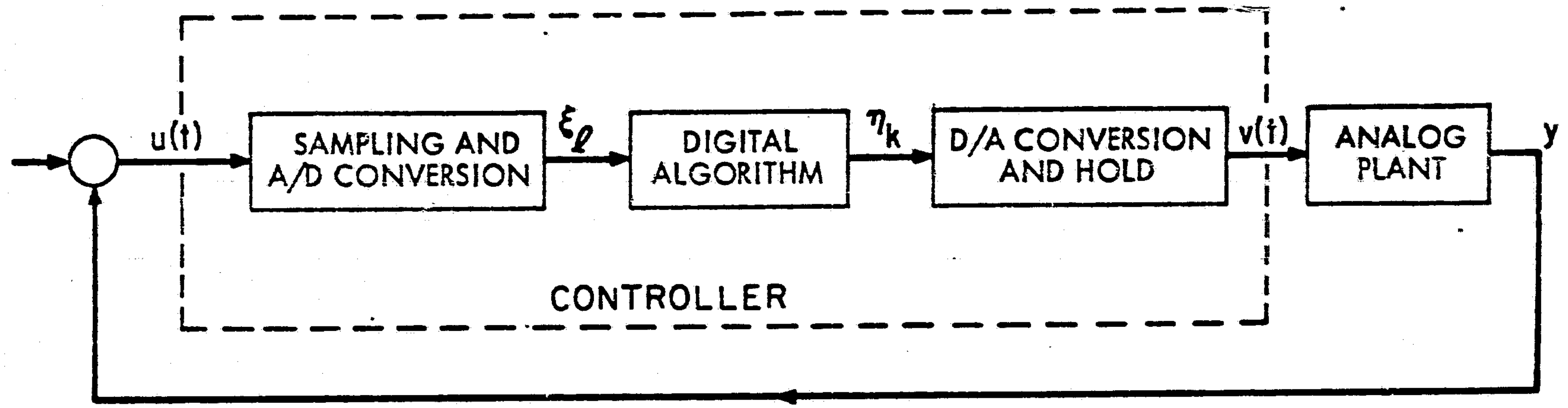

Figure 4. Typical Digitally-Implemented Control System

"zero-order-hold," but in general it will be selected to achieve broader goals. Some examples are again given in the appendix. Note that the controller is completely characterized by the two matrix-valued functions $f_0(\lambda)$, $g_0(\lambda)$ and by the coefficient matrices $D_{k\ell}$.

Hybrid Operator Representation

Given the above description of a digitally-implemented controller, it is straighuforward (Appendix C, Section 2) to write its input-output operator representation, $\underset{\sim}{G}$, in the terms of an impulse response matrix, $G(t,\theta)$. That is,

$$\underline{v} = \underset{\sim}{G}\underline{u} \tag{43}$$

where $\underline{v}$ and $\underline{u}$ denotes functions on $[0,\infty)$ related by the convolution

$$v(t) = \int_0^t G(t,\theta)u(\theta)d\theta \tag{44}$$

with

$$G(t,\theta) = g_0(t-k\tau) \sum_{\ell=1}^{k} D_{k\ell}f_0(\theta-\ell\tau) \tag{45}$$

Here k is understood to be the largest integer less than or equal to t/τ. We will refer to this input-output description of the controller as "the hybrid operator model" or simply as the "hybrid controller." Note that it is a time-varying linear dynamic system characterized by g_0, f_0, and $D_{k\ell}$.

Optimal Hybrid Approximation

Consider now the problem of finding a hybrid operator model $\underset{\sim}{G}(g_0,f_0,D_{k\ell})$ to approximate a continuous-time linear dynamic control law with impulse response matrix

$$\bar{G}(t,\theta) = Ce^{A(t-\theta)}B \quad , \tag{46}$$

where A,B, and C are given system matrices.

Let the approximation criterion be to minimize

$$J = E\left\{\lim_{T\to\infty} \frac{1}{T} \int_0^T ||v(t) - \bar{v}(t)||^2 dt\right\} \tag{47}$$

where $v(t)$ and $\bar{v}(t)$ are the outputs of the hybrid and pure analog controllers respectively, when excited by the same white noise input. Then it is shown in Appendix C, Section 4, that the optimal approximating hybrid controller has the following sampling function:

$$f_0(\lambda) = e^{-A\lambda}B \tag{48}$$

Its corresponding hold function is

$$g_0(\lambda) = Ce^{A\lambda} \quad , \tag{49}$$

and the digital algorithm is

$$D_{k\ell} = M_k d_\ell = e^{AT(k-\ell)} \quad . \tag{50}$$

Moreover, these parameters cause (40) to duplicate (46) exactly everywhere except on the "triangle strip" of Figure 5. Note that the sampling and hold functions (48) - (49) of this optimal hybrid approximation are themselves n-th order dynamic systems, where n is the dimension of A. Hence, the overall hybrid controller can be visualized as shown in Figure 6.

As indicated earlier, the significance of the above result is not the optimal structure in Figure 6 itself (after all, the sampling and hold functions are quite complex, each literally duplicating the analog system), but rather

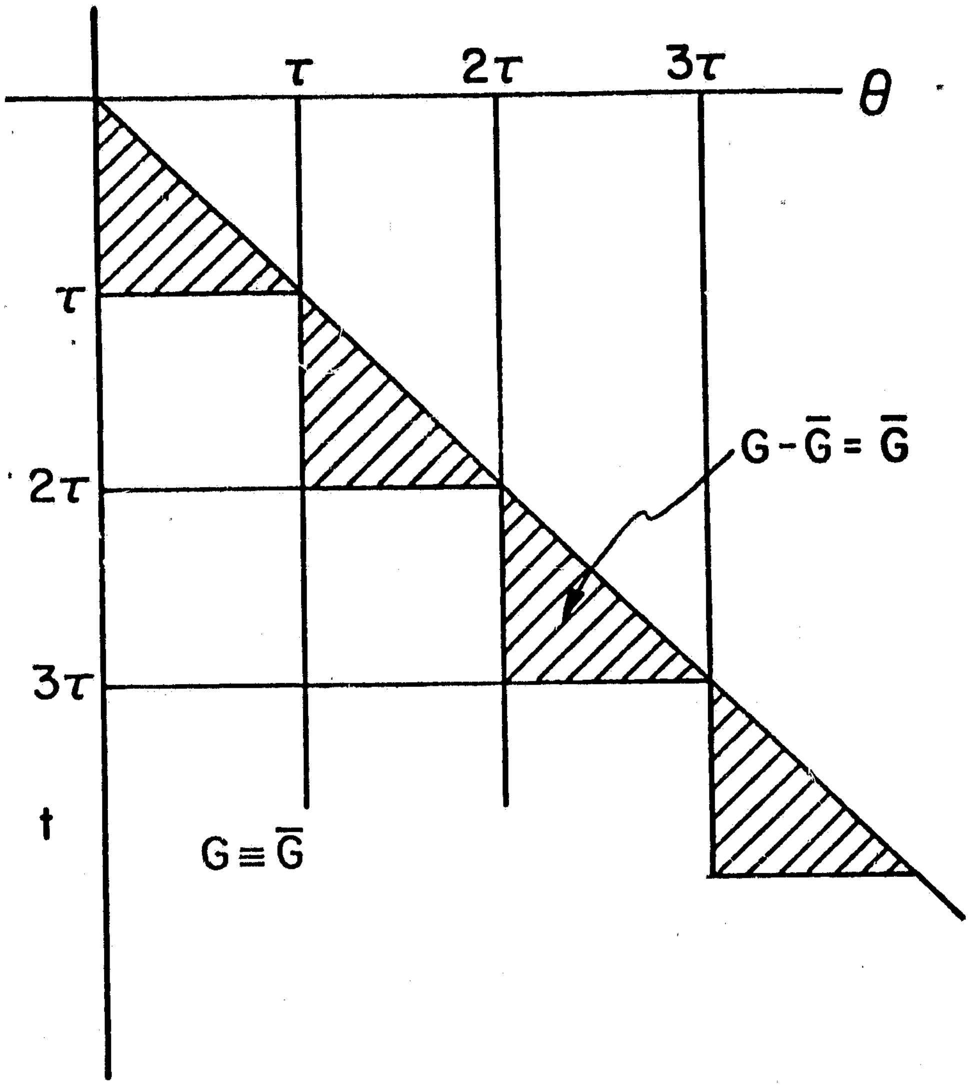

Figure 5. Triangle Strip

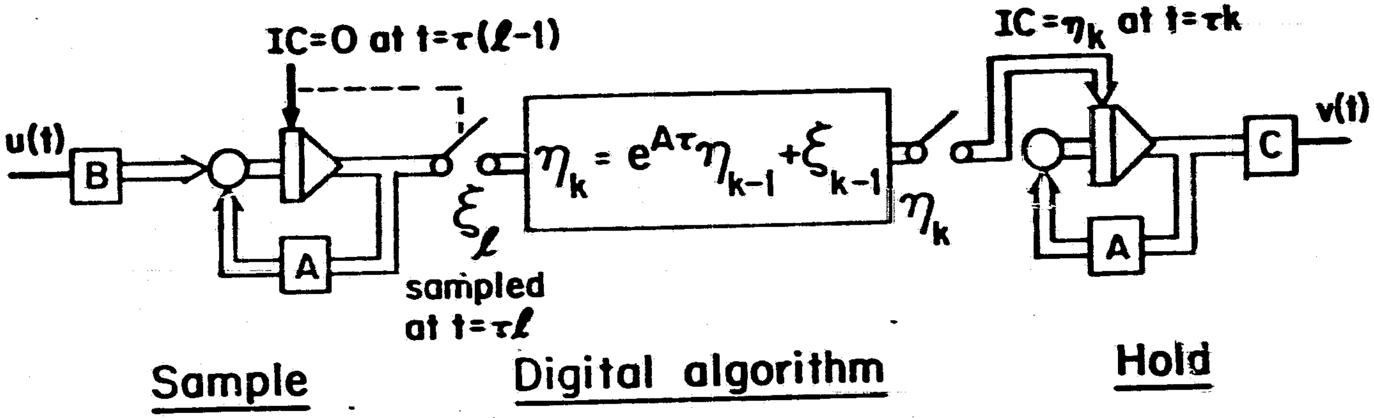

Figure 6. Optimal Approximating Operator

the fact that the inherent hybrid system limitations are so simply and clearly displayed by the triangle strip in Figure 5. It follows from this figure that the minimum approximation error is given by the error operator

$$\underline{e} = (\underset{\sim}{G}-\underset{\sim}{\bar{G}})\underline{u} \tag{51}$$

where $\underset{\sim}{G}-\underset{\sim}{\bar{G}}$ has the impulse response representation

$$e(t) = \int_{kT}^{t} \bar{G}(t-\theta)u(\theta)d\theta \tag{52}$$

Qualitatively therefore, the hybrid system suffers an inherent time varying "data lapse" with a maximum duration of T seconds (average $T/2$), and with data weighting propoortional to the desired impulse response, $\bar{G}$. Hence, both the nominal function $\bar{G}$ and the sample time T contribute to the significance of the error. Small errors are assured if $\bar{G}(\lambda)$ is small over the whole interval $0 \le \tau \le T$ and $u(\theta)$ is relatively "smooth." These observations are given further interpretation later.

Extensions and Applications

Two additional research results are developed in Appendix C which demonstrate the utility of the hybrid operator model. One result deals with constrained optimization of criterion (47), subject to fixed sample and hold structures, and the second deals with error bounds for expression (51).

Constrained Optimization

This result provides optimal approximating hybrid operators which best match a given analog system when the sampling and/or hold circuits are prespecified to take certain (simple) fixed forms. The major results are as follows:

Let
$$\bar{G}(t,\theta) = \bar{H}(t)\bar{S}(\theta) \tag{53}$$

with either (i) $\bar{H}(t_1+t_2) = \bar{H}(t_1)\bar{M}(t_2)$ or

(ii) $\bar{S}(t_1+t_2) = \bar{d}(t_1)\bar{S}(t_2)$ (54)

<u>Fixed Sampler Result</u> (using property (i)):

$f_0(\lambda)$ given, yields

$g_0(\lambda) = \bar{H}(\lambda)$ and

$D_{k\ell} = \bar{M}(k\tau)d_\ell$ with

$$d_\ell = \left[\int_{-\tau}^{0} \bar{S}(\lambda+\tau\ell)f_0^T(\lambda)\,d\lambda\right]\left[\int_{-\tau}^{0} f_0(\lambda)\; f_0^T(\lambda)\,d\lambda\right]^{-1} \tag{55}$$

<u>Fixed Hold Result</u> (using property ii):

$g_0(\lambda)$ given, yields

$f_0(\lambda) = \bar{S}(\lambda)$ and

$D_{k\ell} = M_k\bar{d}(\tau\ell)$ with

$$M_k = \left[\int_0^\tau g_0^T(\lambda)g_0(\lambda)\,d\lambda\right]^{-1}\left[\int_0^\tau g_0^T(\lambda)\; \bar{H}(\lambda+k\tau)\,d\lambda\right] \tag{56}$$

<u>Fixed Sampler and Fixed Hold Result:</u>

$f_0(\lambda)$, $g_0(\lambda)$ both given, yields

$D_{k\ell} = M_k d_\ell$ with M_k and d_ℓ as defined in (16) and (17) (57)

These expressions define optimal digital algorithms and sample or hold functions

under various fixed structure assumptions. In general, their approximation errors will no longer be zero outside the triangle strip. The nature and significance of these added errors remain to be evaluated.

An interesting application of formulas (55) is carried out in Appendix A for the desired nominal system

$$\bar{G}(t,\theta) = e^{A(t-\theta)} = (e^{At})(e^{-A\theta}) \tag{58}$$

with a fixed, nearly impulsive sampling operation

$$f_0(\lambda) = \begin{cases} \frac{1}{\varepsilon} I & -\varepsilon \leq \lambda \leq 0 \\ 0 & \text{elsewhere} \end{cases} \tag{59}$$

The optimal hold is found to be

$$g_0(\lambda) = e^{At} \tag{60}$$

and the corresponding digital algorithm is

$$D_{k\ell} = e^{Ak(\;-\ell)} \int_{-\varepsilon}^{0} e^{-A\lambda} d\lambda \tag{61}$$

The fact to note here is that $D_{k\ell}$ tends to zero as ε becomes small. This is counterintuitive, at first, until we recall that impulsive sampling yields infinite function space norms. $D_{k\ell}$ must tend to zero in order to preserve a finite-gain hybrid approximation of $\bar{G}$. This again highlights the weaknesses associated with pure sample data system representation and with impulsive sampling assumptions.

Error Bounds

The second additional line of research in Appendix C deals with bounds for the inherent approximation errors of the optimal hybrid operator in Figure 6.

This research is motivated by the practical desire to include hybrid operators within the class of systems which can be handled by the stability robustness theory of Safonov, Doyle, Sandell, and Stein [3,13,14,15]. One of the basic results of this theory is the following: A nominally stable feedback system with nominal return difference operator $I + \underset{\sim}{G}$ remains stable under additive perturbations $\underset{\sim}{G} + \Delta\underset{\sim}{G}$ if the perturbations satisfy [14]

$$||(I+\underset{\sim}{G})^{-1}||\ ||\Delta\underset{\sim}{G}|| < 1 \tag{62}$$

This is a special version of equation (9) in Section 2. If $\underset{\sim}{G}$ and $\Delta\underset{\sim}{G}$ are time invariant linear systems with transfer functions $G(s)$ and $\Delta G(s)$, condition (62) is also often written in the form [16]

$$\underline{\sigma}[I+G(j\omega)] < \bar{\sigma}[\Delta G(j\omega)] \qquad \text{for all } \omega \tag{63}$$

where $\underline{\sigma}$ and $\bar{\sigma}$ denote maximum and minimum singular values of their respective matrices.

These stability-robustness results are relevant to our present study of hybrid systems because they provide a way to assess the consequences of hybrid approximation errors. Specifically, if we think of $\underset{\sim}{G}$ as $\tilde{\underset{\sim}{G}}$ (the nominal analog system being approximated) and $\Delta\underset{\sim}{G}$ as the approximation error operator due to digital implementation, (equation (51)) then (62) and (63) provide a way to assess the impact of hybrid approximations on the stability property. In this sense, hybrid errors play exactly the same role as other uncertainties which are associated with the nominal analog system. In fact, if other uncertainties are "large" compared with $\Delta\underset{\sim}{G}$ of (51), then the internal digital nature of the hybrid controller becomes inconsequential. Moreover, it should then be possible to relax (simplify) some of its parameters (samplers, holds, sample rates,

etc.) at the expense of increasing $\Delta \underset{\sim}{G}$. Clearly, simple tight bounds for $||\Delta \underset{\sim}{G}||$ will play a critical role in making these analyses and simplifications possible.

To date, only the following conservative bound is available for ΔG (Appendix C, Section 5):

$$||\Delta \underset{\sim}{G}|| \leq \max_{0 \leq \theta \leq \tau} \bar{\sigma}[G(\theta)]\tau \tag{64}$$

This bound is merely the maximum singular value of $G(t-\theta)$ on the interval $k\tau \leq \theta \leq t$, scaled by τ. The τ-dependence makes it immediately useful as a coarse selection criterion for maximum tolerable sample periods. It tends to be conservative, however. A third order hybrid controller illustration in Appendix C, for example, violates (62) with (64) at τ=0.36 sec. Actual instability does not occur until τ reaches 0.54 sec. Another limitation of (64) is that it does not provide frequency dependent bounds for use in (63). Much tighter bounds should be possible if the frequency content of signals is taken into account. This question forms an important area for future research.

9. CONCLUSIONS

This report has summarized research accomplishments achieved under NASA Research Grant NSG-1312. The overall objectives of this research were to analyze the basic robustness properties of linear-quadratic sample-data regulators and to explore the suitability of these regulators as tools for digital control system design.

The major conclusion of the research is that sample-data LQ regulators are fundamentally inferior to their continuous time counterparts in the sense that their robustness properties are not as good. They share this limitation with continuous LQG designs. In both cases, however, the continuous time properties can be recovered asymptotically by increasing sample rates and by filter redesign, respectively. The research also accomplished important new developments in stability theory, multivariable frequency domain analysis, and mathematical representation of digitally implemented (hybrid) control systems.

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APPENDIX A

ROBUSTNESS OF LQ-OPTIMAL SAMPLED-DATA CONTROL SYSTEM: A SUMMARY OF NEW RESULTS

P. K. Wong
M. Athans

This appendix presents the discrete time version of the continuous-time feedback robustness results for LQ-design reported in [10] (Theorems 1 and 2) and documents a new result for robustness of sampled-data control systems under LQ-design to changes in the sampling rate (Theorem 3).

A. Sampled-data System

Given the system $\dot{x} = Ax + Bu$ $\qquad x \in \mathbb{R}^n, \quad u \in \mathbb{R}^m$ (A1)

We have the following sampled-data model (see Fig. 1):

$$x_{(k+1)\Delta} = A_\Delta x_{k\Delta} + B_\Delta u_{k\Delta} \tag{A2}$$

where

$$\left.\begin{aligned} A_\Delta &\triangleq e^{A\Delta} \\ B_\Delta &\triangleq M_\Delta B \\ M_\Delta &\triangleq \left(\int_0^\Delta e^{A\tau} d\tau\right) \\ &= A^{-1}(e^{A\Delta} - I) \text{ if } A^{-1} \text{ exists} \end{aligned}\right\} \tag{A3}$$

$$\begin{aligned} \Delta &= \text{sampling period} \\ 1/\Delta &= \text{sampling rate} \end{aligned} \tag{A4}$$

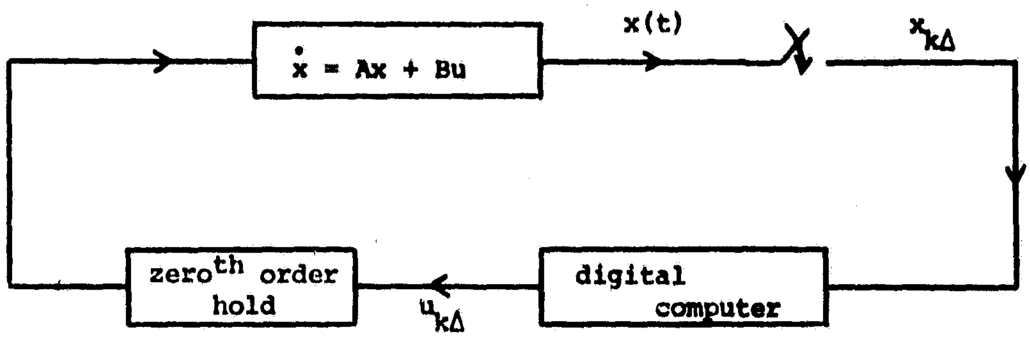

Figure 1

<u>Definition</u>

A_Δ is discrete-time-sense (DT) stable if

$$|\lambda_i(A_\Delta)| < 1 \qquad \forall \text{ eigenvalues } \lambda_i \text{ of } A_\Delta \tag{A5}$$

The following (Lyapunov) results will be of use in deriving the results for the rest of the report:

<u>Discrete-time Stability Theorem</u> (Lyapunov):

Suppose $\exists$ $K > \underline{0}$ and $Q > \underline{0}$ such that

$$K = A^T KA + Q$$

Then A is (DT)-stable.

For the rest of Section A of this report, the notation A_Δ and B_Δ shall be used to denote general discrete-time system parameters i.e., they need not satisfy (A3); the results of Theorems 1 and 2 are valid for any linear discrete-time system, not just those which are sampled-data system models.

LQ-design (DT) - Summary of known results

Problem

$$\min_{\{u_k\}} \sum_{k=0}^{\infty} \left(x_{k+1}^T \, Q \, x_{k+1} + u_k^T \, R \, u_k \right) = J \qquad (Q \geq 0, \; R > 0) \tag{A6}$$

$$\text{s.t. to } x_{k+1} = A_\Delta x_k + \underbrace{M_\Delta B}_{B_\Delta} u_k \qquad \begin{array}{l} (A_\Delta, \; B_\Delta) \text{ stabilizable} \\ (Q^{1/2}, \; A_\Delta) \text{ detectable} \end{array} \tag{A7}$$

Result

$$u_k^{*} = G_\Delta^T \, x_k \tag{A8}$$

$$G_\Delta^T = -(R + B_\Delta^T K \, B_\Delta)^{-1} \, B_\Delta^T K_\Delta A_\Delta \tag{A9}$$

$$K_\Delta = A_\Delta^T \{K_\Delta - K_\Delta B_\Delta (R + B_\Delta^T K_\Delta B_\Delta)^{-1} B_\Delta^T K_\Delta\} \; A_\Delta + Q \tag{A10}$$

$$\Longleftrightarrow \quad K_\Delta = (A_\Delta + B_\Delta G_\Delta^T)^T K_\Delta \; (A_\Delta + B_\Delta G_\Delta^T) + Q + G_\Delta R_\Delta G_\Delta^T \tag{A11}$$

$$\Longleftrightarrow \quad K_\Delta = A_\Delta^T K_\Delta A_\Delta + Q - G_\Delta (R + B_\Delta^T K_\Delta B_\Delta) G_\Delta^T \tag{A12}$$

$$\Longleftrightarrow \quad \hat{K}_\Delta = A_\Delta^T \hat{K}_\Delta A_\Delta + \hat{Q}_\Delta - M_\Delta^T G_\Delta (R + B_\Delta^T \hat{K}_\Delta B) G_\Delta^T \; M_\Delta \tag{A13}$$

$$\text{where} \quad \hat{K}_\Delta \equiv M_\Delta^T K_\Delta M_\Delta \tag{A14}$$

$$\hat{Q}_\Delta \equiv M_\Delta^T Q_\Delta M_\Delta \tag{A15}$$

$$\Longleftrightarrow \quad \hat{K}_\Delta = (A_\Delta + B G_\Delta^T M_\Delta)^T \; \hat{K}_\Delta (A_\Delta + B G_\Delta^T M_\Delta) + \hat{Q}_\Delta + M_\Delta^T G_\Delta R G_\Delta^T M_\Delta \tag{A16}$$

In what follows we shall assume that $Q > 0$ (A17)

Theorem 1 (Discrete-time LQ-gain margin property)

$(A_\Delta + B_\Delta \Lambda G_\Delta^T)$ is stable (DT) if:

$$(G_\Delta^T Q^{-1} G_\Delta)^{-1} + \Lambda^T R \Lambda > (\Lambda^T - I)(R + B_\Delta^T K_\Delta B_\Delta)(\Lambda - I) \tag{A18}$$

or equivalently,

$$\left((G_\Delta^T Q^{-1} G_\Delta)^{-1} + R(B_\Delta^T K_\Delta B_\Delta)^{-1} R + R\right) >$$

$$\left(\Lambda^T - I - R(B_\Delta^T K_\Delta B_\Delta)^{-1}\right)(B_\Delta^T K_\Delta B_\Delta)\left(\Lambda - I - (B_\Delta^T K_\Delta B_\Delta)^{-1} R\right) \qquad \text{(A19)}$$

▨

Remark

From equation (A19), in Theorem 1 it is obvious that there is both an upper bound as well as a lower bound on the values of the admissible Λ. This is most transparently demonstrated in the case when there is only a single control-input, so that Λ becomes a scalar in this case:

Corollary 1.1

(for the case of a single-control input LQ-design)

$\left[A_\Delta + b_\Delta \alpha g_\Delta^T\right]$ is (DT)-stable if

$$(1 + \frac{r}{m}) - \sqrt{\frac{1}{m}\left(\frac{1}{w} + r\ (1 + \frac{r}{m})\right)} < \alpha < (1 + \frac{r}{m}) + \sqrt{\frac{1}{m}\left(\frac{1}{w} + r(1 + \frac{r}{m})\right)} \qquad \text{(A20)}$$

where $m \triangleq b_\Delta^T K_\Delta b_\Delta$

$w \triangleq g_\Delta^T Q^{-1} g_\Delta$

▨

Theorem 1 can be re-stated in a different way which shows explicitly the range of admissible Λ, by simply generalizing the above 'square-rooting' procedure used for the single-input case to the multi-input situation:

Corollary 1.2

$(A_\Delta + B_\Delta \Lambda G_\Delta^T)$ is (DT)-stable $\forall$

$$\Lambda = (I + W_\Delta^{-1}R) \pm (M_1)^{\frac{1}{2}}\Sigma^{\frac{1}{2}}(W_\Delta)^{-\frac{1}{2}}$$

$$\text{for all } \Sigma = \Sigma^T \quad \text{s.t.} \quad 0 \leq \Sigma < I$$

where

$$M_1 \triangleq \left[(G_\Delta^T Q^{-1} G_\Delta)^{-1} + R\, W_\Delta^{-1} R + R\right]$$

$$W_\Delta \triangleq (B_\Delta^T K_\Delta B_\Delta)$$

<u>Theorem 2</u> (General Discrete-time LQ Gain Robustness)

$(A_\Delta + B_\Delta(FG_\Delta^T + \delta G^T))$ is DT-stable $\forall\ \delta G \in R(Q^{-1}G_\Delta)^1$ and $F \in \mathbb{R}^{m\times m}$ such that

$$\delta G^T Q^{-1} \delta G < \left[W_\Delta + (W_\Delta(I-F) + R)Z^{-1}((I-F^T)W_\Delta + R)\right]^{-1} \tag{A21}$$

where

$$Z \triangleq (G_\Delta^T Q^{-1} G_\Delta)^{-1} + F^T R F - (I-F^T)(W_\Delta + R)(I-F) > \underline{0}$$

and

$$W_\Delta \triangleq B_\Delta^T K_\Delta B_\Delta \tag{A22}$$

<u>Remark</u>

Theorem 2 is the discrete-time version of Corollary 2.1 in Wong, Stein and Athans for the continuous case. Unlike the continuous-time result, the discrete-time version is much more complicated and is probably of little computational usefulness.

B. <u>Sampling-time Robustness of LQ-design</u>

Given the system $\dot{x} = Ax + Bu$ (B1)

Suppose we sample the system state at rate $(^1/_\Delta)$. The equivalent sampled-data model of the system is:

$$x_{(k+1)\Delta} = A_\Delta x_{k\Delta} + B_\Delta u_{k\Delta} \tag{B2}$$

$$A_\Delta = e^{A\Delta}$$

$$B_\Delta = M_\Delta B = \left(\int_0^\Delta e^{A\tau} d\tau\right) B$$

If we choose a discrete-time LQ-design to stabilize the sampled data model (B2), then the closed-loop system $(A_\Delta + B_\Delta G_\Delta^T)$ is stable, where G_Δ^T is the optimal gain computed for the sampled-data model when the sampling rate is $^1/_\Delta$ and for the cost-weightings Q and R.

Suppose now we change the sampling rate to $(\frac{1}{\Delta+\Delta'})$; the corresponding sampled-data model becomes

$$x_{(k+1)(\Delta+\Delta')} = A_{\Delta+\Delta'}\, x_{k(\Delta+\Delta')} + B_{\Delta+\Delta'}\, u_{k(\Delta+\Delta')} \tag{B3}$$

If we <u>do not</u> change the gains G_Δ^T computed previously, the new closed-loop system at the new sampling rate becomes

$$(A_{\Delta+\Delta'} + B_{\Delta+\Delta'}\, G_\Delta^T) \tag{B4}$$

which of course is not necessarily stable. The problem we want to pose is: for what range of Δ' would (B4) remain stable?

We have the following sufficiency result:

<u>Theorem 3</u> (Sampling-rate robustness of LQ-design)

$(A_{\Delta+\Delta'} + B_{\Delta+\Delta'}\, G_\Delta^T)$ is stable if

$$K_\Delta - e^{A^T\Delta'}(K_\Delta - Q)e^{A\Delta'} + (e^{A\Delta'} + M_{\Delta'}M_\Delta^{-1})^T G_\Delta R G_\Delta^T (e^{A\Delta'} + M_{\Delta'}M_\Delta^{-1})$$

$$> (M_{\Delta'}M_\Delta^{-1})^T G_\Delta (R + B_\Delta^T K_\Delta B_\Delta) G_\Delta^T (M_{\Delta'}M_\Delta^{-1}) \tag{B5}$$

▨

Remark

$$M_{\Delta'}M_\Delta^{-1} = (I - e^{A\Delta'})(I - e^{A\Delta})^{-1} \text{ if } A^{-1} \text{ exists}$$

Remark

It is not obvious what physical interpretation can be made of the expression (B5); some numerical examples will be worked out to gain insight into the meaning of (B5) in future research.

Remark

The case when $\Delta' = \Delta$ (i.e. doubling of sampling period) is a particularly simple special case of Theorem 3:

Corollary (Robustness to doubling of sampling period)

$(A_{2\Delta} + B_{2\Delta}G_\Delta^T)$ is (DT)-stable if

$$Q + e^{A^T\Delta} Q\, e^{A\Delta} + (e^{A^T\Delta} + I) G_\Delta R G_\Delta^T (e^{A\Delta} + I) > 2\, G_\Delta (R + B_\Delta^T K_\Delta B_\Delta) G_\Delta^T \tag{B6}$$

▨

APPENDIX

Proof of Theorem 1

The proof of Theorem 1 follows immediately from Theorem 2, as (A21) is automatically satisfied for $\delta G \equiv 0$, and we need only to ensure that (A22) holds, but (A18) is just (A22) by substituting $\Lambda = F^T$. To show that (A18) is equivalent to (A19), we just have to "complete square" by appropriately factoring Λ. ▨

(algebraic details: let $G_\Delta^T Q G_\Delta \overset{\Delta}{=} W_o$, $B_\Delta^T K_\Lambda B_\Delta = M$

Then (A18) $\iff$ $W_o^{-1} + \Lambda^T R\Lambda - [\Lambda^T(R+M)\Lambda - (R+M)\Lambda - \Lambda^T(R+M) + (R+M)] > 0$

$$\iff W_o^{-1} + \Lambda^T(R+M) + (R+M)\Lambda - (R+M) - \Lambda^T M\Lambda > 0$$

$$\iff W_o^{-1} - (R+M) - (\Lambda^T M\Lambda - \Lambda^T(R+M) + (R+M)\Lambda) > 0$$

$$\iff W_o^{-1} - (R+M) + (R+M)M^{-1}(R+M) - [\Lambda^T M - (R+M)]M^{-1}[M\Lambda - (R+M)] > 0$$

$$\iff W_o^{-1} - (R+M) + RM^{-1}R + 2R + M - [\Lambda^T - (I + RM^{-1})]M[\Lambda - (I+M^{-1}R)] > 0$$

$$\iff W_o^{-1} + R + RM^{-1}R > [\Lambda^T - (I + RM^{-1})]M[\Lambda - (I+M^{-1}R)]$$

<u>Proof of Corollary 1.2</u>

The proof utilizes the following lemma:

<u>Lemma 0</u>

$$\{H^T | H^T M_2 H < M_1 \quad \text{where } M_1 > 0,\ M_2 > 0\}$$

$$= \{H^T | H^T = \pm M_1^{\frac{1}{2}}\Lambda^{\frac{1}{2}}(M_2^{\frac{1}{2}})^{-1}, \quad \forall\ \underline{0} \leq \Lambda < I\}$$

<u>Proof</u>

$$H^T = \pm M_1^{\frac{1}{2}}\Lambda^{\frac{1}{2}}(M_2^{\frac{1}{2}})^{-1}$$

$$\Rightarrow \quad H^T M_2 H = M_1^{\frac{1}{2}}\Lambda M_1^{\frac{1}{2}} < M_1^{\frac{1}{2}}M_1^{\frac{1}{2}} = M_1$$

<u>Proof of Corollary 2.1:</u>

Let $H^T = \Lambda - (I + W_\Delta^{-1}R)$ in the above lemma and substituting appropriately for M_1, M_2.

Proof of Theorem 2

(Assume without loss of generality that $[G_\Delta \delta G]$ is of full rank in what follows)

We have

$$K_\Delta = A_\Delta^T K_\Delta A_\Delta + Q - G_\Delta (R + B_\Delta^T K_\Delta B_\Delta) G_\Delta^T \tag{2.1}$$

so

$$K_\Delta = (A_\Delta + B_\Delta (F G_\Delta^T + \delta G^T))^T K_\Delta (A_\Delta + B_\Delta (F G_\Delta^T + \delta G^T)) + \tilde{Q} \tag{2.2}$$

where

$$\begin{aligned}
\tilde{Q} \triangleq\ & Q - G_\Delta (R + W_\Delta) G_\Delta^T - \left[(G_\Delta F^T + \delta G) B_\Delta^T K_\Delta A_\Delta + A_\Delta^T K_\Delta B_\Delta (F G_\Delta^T + \delta G^T) \right] \\
& - (G_\Delta F^T + \delta G) B_\Delta^T K_\Delta B_\Delta (F G_\Delta^T + \delta G^T) \\
=\ & Q - G_\Delta [R + W_\Delta + F^T W_\Delta F] G_\Delta^T \\
& - (G_\Delta [F^T B_\Delta^T K_\Delta A_\Delta + F^T W_\Delta \delta G^T] + [A_\Delta^T K_\Delta B_\Delta F + \delta G W_\Delta F] G_\Delta^T) \\
& - (\delta G [B_\Delta^T K_\Delta A_\Delta] + [A_\Delta^T K_\Delta B_\Delta] \delta G^T) \\
& - \delta G\, W_\Delta \delta G^T \\
=\ & Q - G_\Delta [R + W_\Delta + F^T W_\Delta F] G_\Delta^T \\
& + G_\Delta [F^T (R + W_\Delta) G_\Delta^T - F^T W_\Delta \delta G^T] + [G_\Delta (R + W_\Delta) F - \delta G\, W_\Delta F] G_\Delta^T \\
& + \delta G (R + W_\Delta) G_\Delta^T + G_\Delta (R + W_\Delta) \delta G^T - \delta G_\Delta W_\Delta \delta G^T \\
=\ & Q + G_\Delta [F^T (R + W_\Delta) + (R + W_\Delta) F - (R + W_\Delta) - F^T W_\Delta F] G_\Delta^T \\
& + G_\Delta [R + W_\Delta - F^T W_\Delta] \delta G^T + \delta G [R + W_\Delta - W_\Delta F] G_\Delta^T \\
& - \delta G_\Delta W_\Delta \delta G^T \\
=\ & Q + [G_\Delta\ \delta G] \underbrace{\left[\begin{array}{c:c} F^T R F - (F^T - I)(R + W_\Delta)(F - I) & R - (F^T - I) W_\Delta \\ \hdashline R - W_\Delta (F - I) & -W_\Delta \end{array} \right]}_{\triangleq M} \begin{bmatrix} G_\Delta^T \\ \delta G^T \end{bmatrix}
\end{aligned} \tag{2.3}$$

Since $Q > \underline{0}$, we have

$$\tilde{Q} > 0 \iff Q^{-1} + Q^{-1}[G_\Delta \quad \delta G] \, M \begin{bmatrix} G_\Delta^T \\ \delta G^T \end{bmatrix} Q^{-1} > 0 \tag{2.4}$$

$$\iff \begin{bmatrix} G_\Delta^T \\ \delta G^T \end{bmatrix} \left(Q^{-1} + Q^{-1} \, [G_\Delta \quad \delta G] \, M \begin{bmatrix} G_\Delta^T \\ \delta G^T \end{bmatrix} Q^{-1} \right) [G_\Delta \quad \delta G_\Delta] > 0$$

(see Lemma 1 in Appendix of Wong, Stein, Athans)

$$\iff \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} + \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} M \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} > \underline{0} \tag{2.5}$$

where $\quad X_1 \triangleq G_\Delta^T Q^{-1} G_\Delta$

$$X_2 \triangleq \delta G^T Q^{-1} \delta G \quad \text{and} \quad \delta G \in R(Q^{-1} G_\Delta) \tag{2.6}$$

$$\iff \begin{bmatrix} X_1 + X_1 M_1 X_1 & X_1 M_{12} X_2 \\ X_2 M_{21} X_1 & X_2 M_2 X_2 + X_2 \end{bmatrix} > \underline{0} \tag{2.7}$$

$$\iff \begin{cases} X_1 + X_1 M_1 X_1 > \underline{0} & (2.8) \\ X_2 + X_2 \{M_2 - M_{21} X_1 (X_1 M_1 X_1 + X_1)^{-1} X_1 M_{12}\} X_2 > \underline{0} & (2.9) \end{cases}$$

$$(2.8) \iff X_1^{-1} + M_1 > \underline{0}$$

$$\iff Z \triangleq (G_\Delta^T Q^{-1} G_\Delta) + F^T R F - (F^T - I)(R + W_\Delta)(F - I) > \underline{0} \tag{2.10}$$

(2.9) and (2.10)

$$\Leftrightarrow \quad X_2 + X_2 \{M_2 - M_{21}Z^{-1}M_{12}\}X_2 > \underline{0}$$

$$\Leftrightarrow \quad X_2^{-1} + \{M_2 - M_{21}Z^{-1}M_{12}\} > \underline{0}$$

$$\Leftrightarrow \quad X_2^{-1} + \{-W_\Delta - (R + W_\Delta(I-F))Z^{-1} (R + (I-F^T)W_\Delta)\} > 0$$

$$\Leftrightarrow \quad X_2^{-1} > [W_\Delta + (R + W_\Delta(I-F))Z^{-1} (R + (I-F^T)W_\Delta)] > \underline{0}$$

$$X_2 < [W_\Delta + (R + W_\Delta(I-F))Z^{-1} (R + (I-F^T)W_\Delta)] \tag{2.11}$$

Thus we have shown that

$$(2.10) \text{ and } (2.11) \Leftrightarrow \tilde{Q} > \underline{0} \Rightarrow (A_\Delta + B_\Delta(F\, G_\Delta^T + \delta G^T)) \text{ is stable (DT)}$$

from (2.2)

Q.E.D.

<u>Proof of Theorem 3</u>

The proof of Theorem 3 is facilitated by the following lemmas:

<u>Lemma 1</u> A_Δ commutes with $M_{\Delta'}$ and $M_{\Delta'}^{-1}$ $\forall\ \Delta$ and Δ' ▨

<u>Proof</u>

$$e^{A\Delta}\int_0^{\Delta'} e^{A\tau}d\tau = \int_0^{\Delta'} e^{A\tau}d\tau\; e^{A\Delta}$$

$$e^{A\Delta}\left(\int_0^{\Delta'} e^{A\tau}d\tau\right)^{-1} = \left(e^{-A\Delta}\right)^{-1}\left(\int_0^{\Delta'} e^{A\tau}d\tau\right)^{-1} = \left(\int_0^{\Delta'} e^{A\tau}d\tau\; e^{-A\Delta}\right)^{-1}$$

$$= \left(\int_0^{\Delta'} e^{A\tau}d\tau\right)^{-1} e^{A\Delta}$$

▨

Lemma 2

$$(A_\Delta + B_\Delta G^T) \text{ is stable } \Longleftrightarrow (A_\Delta + B\, G^T M_\Delta) \text{ is stable.}$$ ▨

Proof

$$(A_\Delta + B_\Delta G^T) \text{ stable}$$

$$\Longleftrightarrow \; M_\Delta^{-1}(A_\Delta + B_\Delta G^T)M_\Delta \text{ stable}$$

$$= M_\Delta^{-1}A_\Delta M_\Delta + B\, G^T M_\Delta$$

$$= M_\Delta^{-1}M_\Delta A_\Delta + B\, G^T M_\Delta \qquad (\because A_\Delta M_\Delta = M_\Delta A_\Delta)$$

$$= A_\Delta + B\, G^T M_\Delta$$ ▨

Lemma 3

$$M_{\Delta+\Delta'} = M_\Delta + A_\Delta M_{\Delta'} = M_{\Delta'} + A_{\Delta'} M_\Delta$$

Proof

$$M_{\Delta+\Delta'} = \int_o^{\Delta+\Delta'} e^{A\tau}d\tau = \int_o^{\Delta} e^{A\tau}d\tau + \int_\Delta^{\Delta+\Delta'} e^{A\tau}d\tau$$

$$= M_\Delta + e^{A\Delta}\int_\Delta^{\Delta+\Delta'} e^{A(\tau-\Delta)}d\tau$$

$$= M_\Delta + e^{A\Delta} M_{\Delta'}$$ ▨

Proof of Theorem 3

We have $(A_{\Delta+\Delta'} + B_{\Delta+\Delta'} G_{\Delta}^{T})$ stable (DT)

$$\Leftrightarrow (A_{\Delta+\Delta'} + B\, G_{\Delta}^{T}\, M_{\Delta+\Delta'}) \text{ stable (DT) from Lemma 2} \tag{3.1}$$

$$= (A_{\Delta} + B\, G_{\Delta}^{T}\, M_{\Delta+\Delta'} A_{\Delta'}^{-1}) A_{\Delta'}$$

$$= (A_{\Delta} + B\, G_{\Delta}^{T} (M_{\Delta'} + M_{\Delta} A_{\Delta'}) A_{\Delta'}^{-1}) A_{\Delta'} \quad \text{from Lemma 3}$$

$$= (A_{\Delta} + B\, G_{\Delta}^{T}\, (M_{\Delta} + M_{\Delta'} A_{\Delta'}^{-1})) A_{\Delta'} \tag{3.2}$$

Now

$$K_{\Delta} = A_{\Delta}^{T} K_{\Delta} A_{\Delta} + Q - G_{\Delta} (R + B_{\Delta}^{T} K_{\Delta} B_{\Delta}) G_{\Delta}^{T}$$

$$\Leftrightarrow \hat{K}_{\Delta} = A_{\Delta}^{T} \hat{K}_{\Delta} A_{\Delta} + \hat{Q}_{\Delta} - M_{\Delta}^{T} G_{\Delta} (R + B_{\Delta}^{T} \hat{K}_{\Delta} B) G_{\Delta}^{T} M_{\Delta} \qquad \text{from (A12)}$$

where $\hat{K}_{\Delta} = M_{\Delta}^{T} K_{\Delta} M_{\Delta}$ to (A15)

$\hat{Q}_{\Delta} = M_{\Delta}^{T} Q_{\Delta} M_{\Delta}$

so

$$\hat{K}_{\Delta} = \Big[A_{\Delta} + B\, G_{\Delta}^{T} \underbrace{(M_{\Delta} + M_{\Delta'} A_{\Delta'}^{-1})}_{\triangleq\, P}\Big]^{T} \hat{K}_{\Delta} \Big[A_{\Delta} + B\, G_{\Delta}^{T} (M_{\Delta} + M_{\Delta'} A_{\Delta'}^{-1})\Big] + \tilde{Q}_{\Delta} \tag{3.3}$$

where

$$\tilde{Q}_{\Delta} \equiv \hat{Q}_{\Delta} + P^{T} G_{\Delta} R_{\Delta} G_{\Delta}^{T} P - (P^{T} - M_{\Delta}) G_{\Delta} W_{\Delta} G_{\Delta}^{T} (P - M_{\Delta}) \tag{3.4}$$

(after some algebraic manipulation)

with $W_{\Delta} \equiv B^{T} \hat{K}_{\Delta} B$

$$(3.3) \Rightarrow A_{\Delta'}^{T} \hat{K}_{\Delta} A_{\Delta'} = A_{\Delta'}^{T} (A_{\Delta} + B\, G_{\Delta}^{T} P)^{T}\, \hat{K}\, (A_{\Delta} + B_{\Delta} G_{\Delta}^{T} P) A_{\Delta'} + A_{\Delta'}^{T} \tilde{Q}_{\Delta} A_{\Delta}$$

$$= (A_{\Delta+\Delta'} + B\, G_{\Delta}^{T} M_{\Delta+\Delta'})^{T}\, \hat{K}_{\Delta}\, \underbrace{(A_{\Delta+\Delta'} + B\, G_{\Delta}^{T} M_{\Delta+\Delta'})}_{\triangleq\, A_c}$$

$$+ A_{\Delta'}^{T} \tilde{Q}_{\Delta} A_{\Delta'} \tag{3.5}$$

$$\Rightarrow \quad \hat{K}_\Delta - A_c^T \hat{K}_\Delta A_c = \hat{K}_\Delta - A_{\Delta'}^T(\hat{K}_\Delta - \tilde{Q}_\Delta)A_{\Delta'} \tag{3.6}$$

since $\hat{K} > \underline{0}$, we have

$$\hat{K}_\Delta - A_c^T \hat{K}_\Delta A_c > \underline{0} \Rightarrow A_c \text{ is (DT) stable} \tag{3.7}$$

Thus (3.6) and (3.7) together $\Rightarrow$

$$\hat{K}_\Delta - A_{\Delta'}^T(\hat{K}_\Delta - \tilde{Q}_\Delta)A_{\Delta'} > 0 \Rightarrow (A_{\Delta+\Delta'} + B\, G_\Delta^T M_{\Delta+\Delta'}) \text{ stable}$$

$$\Leftrightarrow \quad (A_{\Delta+\Delta'} + B_{\Delta+\Delta'} G_\Delta^T) \text{ stable}$$

Now $\quad \hat{K}_\Delta - A_{\Delta'}^T(\hat{K}_\Delta - \tilde{Q}_\Delta)A_{\Delta'} > 0$

$$\Leftrightarrow \quad K_\Delta - A_{\Delta'}^T(K_\Delta - M_\Delta^{T-1}\tilde{Q}_\Delta M_\Delta^{-1})A_{\Delta'} > \underline{0}$$

$$\Leftrightarrow \quad K_\Delta - A_{\Delta'}^T(K_\Delta - Q)A_{\Delta'} + A_{\Delta'}^T M_\Delta^{T-1} P^T G_\Delta R\, G_\Delta^T P\, M_\Delta^{-1} A_{\Delta'}$$

$$> \quad A_{\Delta'}^T M_\Delta^{T-1}(P^T - M_\Delta) G_\Delta W_\Delta G_\Delta^T (P - M_\Delta) M_\Delta^{-1} A_{\Delta'}$$

$$\Leftrightarrow \quad K_\Delta - A_{\Delta'}^T(K_\Delta - Q)A_{\Delta'} + (A_{\Delta'} + M_{\Delta'} M_\Delta^{-1})^T G_\Delta R\, G_\Delta^T (A_{\Delta'} + M_{\Delta'} M_\Delta^{-1})$$

$$> \quad (M_{\Delta'} M_\Delta^{-1})^T G_\Delta (R + B_\Delta^T K_\Delta B_\Delta) G_\Delta^T (M_{\Delta'} M_\Delta^{-1})$$

APPENDIX B

ATTACHMENT 1 (of ESL-SR-835)

GAIN-MARGINS AND STABILITY ROBUSTNESS OF LQG REGULATOR

by

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ABSTRACT

New sufficiency characterizations of the gain-margins of the standard full LQG (Linear-Quadratic-Gaussian) regulator design (which incorporates a Kalman filter in the feedback loop) are presented. These results show that full recovery of LQSF-gain margins can be achieved either through non-divergent filter structure adaptation, or when plant-driving noise that enters through the same channels as the control inputs greatly dominate other noise terms. An explicit sufficiency bound on the gain-margins of LQG-design that varies with a ratio of quadratic forms of the filter error dynamics & the plant dynamics is also presented. These results further clarify the recent work of Doyle, and suggest potential new directions of research.

1. Introduction

As has been demonstrated in recent research (e.g. [1], [2] and [3] and references cited therein), the standard multivariable Linear-Quadratic-State-Feedback regulator design (LQSF) is known to have rather robust stability properties. In particular, as has been shown in [3], the LQSF control design has the following gain-margin property.

If $\underline{u}^*(t)$ is the optimal LQSF control gain-vector, then the closed-loop system plant under the control of $\underline{u}^*(t)$ remains stable for all gain perturbations:

$$\underline{u}^*(t) \longmapsto \underline{\Lambda}(t)\underline{u}^*(t)$$

where

$$\underline{\Lambda}(t) = \begin{bmatrix} \alpha_1(t) & & \underline{0} \\ & \ddots & \\ \underline{0} & & \alpha_n(t) \end{bmatrix} \quad \text{is such that}$$

$$\underline{\Lambda}(t) > \frac{1}{2}(\underline{I} - \underline{X}_o^{-1})$$

where $\underline{X}_o \triangleq (\underline{R}^{1/2}\underline{G}_o^T\underline{Q}^{-1}\underline{G}_o\underline{R}^{1/2})$, $\underline{Q} > \underline{0}$, $\underline{R}$ = diagonal matrix > $\underline{0}$ being the LQ cost weightings, and $\underline{G}_o^T$ is the optimal gain matrix.

That is, LQSF guarantees strictly greater than - 6db. gain reduction & infinite gain margin, regardless of the choice of cost criteria $\underline{Q} > \underline{0}$ and $\underline{R}$ diagonal > $\underline{0}$.

Because of this and other stability robustness properties of the LQSF (see [2],[3] for further details), there has been great interest in the question as to whether the full Linear-Quadratic-Gaussian regulator design, which employs output feedback using a Kalman filter, retains any of these stability robustness properties in general and the gain margin property (GM) stated above in particular. In a short paper entitled "Guaranteed margins for LQG regulators," and carrying an abstract with the single sentence "There aren't any," J.C. Doyle has shown through a simple counter-example that <u>there exists no guaranteed gain margins independent of the choice of cost-criteria & noise characteristics specification</u>. In other words, design-parameter-dependent characterizations of the gain-margins of full LQG-system need to be investigated before one can evaluate the stability robustness of the LQG-methodology.

It is the aim of this report to present preliminary results of our research in investigating the design-parameter-dependent characterization of the gain margins of LQG regulator.

The organization of this paper is as follows. In Section 2, we state our formulation of the full LQG gain margin characterization problem. In Section 3, some useful sufficiency results which we have obtained are reported and their significance discussed. Finally, in Section 4, we present discussion on potential future research directions.

Notations and Definitions

$\underline{A}^T$ denotes the transpose of $\underline{A}$

$R(\underline{H})$ denotes the range space of $\underline{H}$

$R(\underline{H})^{\perp}$ denotes the orthogonal complement of $R(\underline{H})$

If $\underline{Q} \in R^{nxn}$ is positive definite (positive semidefinite), we will writes $\underline{Q} > \underline{0}$ $(\underline{Q} \geq \underline{0})$

If $\underline{Q} \geq \underline{0}$ and $\underline{x}^T \underline{Q} \underline{x} > 0$ for all $\underline{x} \in R(\underline{H})$, $\underline{x} \neq \underline{0}$, we write

$\underline{Q} > \underline{0}\Big|_{R(\underline{H})}$

(i.e. the positive semidefinite matrix $\underline{Q}$ is positive in the range space of $\underline{H}$).

2. Problem Formulation

Given the linear time-invariant dynamic system $(\underline{A}, \underline{B}, \underline{C}^T)$ such that

$$(\underline{A},\underline{B}) \quad \text{is a stabilizable pair} \tag{1}$$

$$(\underline{C}^T,\underline{B}) \quad \text{is a detectable pair} \tag{2}$$

Let $\underline{G}_o^T$ denote the optimal LQSF-control gain for some $\underline{Q} > \underline{0}$ and diagonal $\underline{R} > \underline{0}$, where

$$\underline{KA} + \underline{A}^T\underline{K} - \underline{KBR}^{-1}\underline{B}^T\underline{K} + \underline{Q} = \underline{0} \tag{3}$$

and

$$\underline{G}_o^T \triangleq - \underline{R}^{-1}\underline{B}^T\underline{K} \tag{4}$$

Further, let $\underline{H}_o$ denote the optimal Kalman filter gain for some $\underline{\Xi} > \underline{0}$ and $\underline{\Theta} > \underline{0}$, where

$$\underline{\Sigma}\,\underline{A}^T + \underline{A}\underline{\Sigma} - \underline{\Sigma}\,\underline{C}\,\underline{\Theta}^{-1}\underline{C}^T\,\underline{\Sigma} + \underline{\Xi} = \underline{0} \tag{5}$$

and

$$\underline{H}_o \triangleq -\underline{\Sigma}\,\underline{C}\,\underline{\Theta}^{-1} \tag{6}$$

Then the closed-loop, full LQG-system becomes:

$$\begin{bmatrix} \dot{\underline{x}} \\ \dot{\hat{\underline{x}}} \end{bmatrix} = \begin{bmatrix} \underline{A} & \underline{B}\underline{G}_o^T \\ -\underline{H}_o\underline{C}^T & \underline{A} + \underline{H}_o\underline{C}^T + \underline{B}\underline{G}_o^T \end{bmatrix} \begin{bmatrix} \underline{x} \\ \hat{\underline{x}} \end{bmatrix} + \text{(noise terms)} \tag{7}$$

where

$\underline{x}(\cdot)$ = plant state vector

$\hat{\underline{x}}(\cdot)$ = filter state vector

(we shall ignore the external noise terms in what follows as they are not relevant in subsequent discussion on closed-loop stability).

Suppose now that the optimal feedback $\underline{u}^*(t) = \underline{G}_o^T\underline{x}(t)$ is perturbed:

$$\underline{u}^*(t) \longmapsto \underline{\Lambda}(t)\underline{u}^*(t) \tag{8}$$

where $\underline{\Lambda}(t)$ is a diagonal matrix for all $t \in [0,\infty)$.

The perturbed closed-loop system becomes

$$\begin{bmatrix} \dot{\underline{x}}(t) \\ \dot{\hat{\underline{x}}}(t) \end{bmatrix} = \begin{bmatrix} \underline{A} & \underline{B}\,\underline{\Lambda}(t)\underline{G}_o^T \\ -\underline{H}_o\underline{C}^T & \underline{A} + \underline{B}\,\underline{G}_o^T + \underline{H}_o\underline{C}^T \end{bmatrix} \begin{bmatrix} \underline{x} \\ \hat{\underline{x}} \end{bmatrix} \tag{9}$$

Problem: For what range of $\underline{\Lambda}(t)$, $t \in [0,\infty)$ can we guarantee that the purturbed system (9) is stable in the sense that

$$\begin{bmatrix} \underline{x}(t) \\ \hat{\underline{x}}(t) \end{bmatrix} \to \underline{0} \qquad \text{as } t \to \infty \qquad ?$$

Remark:

The class of perturbation (8) includes the class of all non-dynamic, nonlinear functions:

$$\left.\begin{array}{ll} u_i^*(t) \quad \longmapsto \quad f_i(u_i^*(t),t) & i=1,\ldots,m \\ \text{provided} & \\ f_i(0,t) = 0 & \end{array}\right\} \tag{10}$$

This follows from the simple observation that, given (10), we can define

$$\left.\begin{array}{lll} \lambda_i(t) \triangleq \dfrac{f_i(u_i^*(t),t)}{u_i^*(t)} \ , & & u_i^*(t) \neq 0 \\ \qquad\ \triangleq \text{arbitrary} \ , & & u_i^*(t) = 0 \end{array}\right\} \quad i = 1,\ldots,m \tag{11}$$

and

$$\underline{\Lambda}(t) \triangleq \begin{bmatrix} \lambda_1(t) & & \underline{0} \\ & \ddots & \\ \underline{0} & & \lambda_m(t) \end{bmatrix}$$

Note that the restriction of diagonality on $\underline{\Lambda}(t)$ was made in (8) because of the natural interpretation of $\underline{\Lambda}(t)$ that follows from (10) and (11).

Remark:

We shall first examine the case

$$\underline{\Lambda}(t) \equiv \underline{\Lambda} \quad \text{constant matrix} \tag{12}$$

in what follows. The general time-varying case of $\underline{\Lambda}(t)$ will be covered by a trivial generalization of the time-constant case in a later section. This procedure of presentation not only simplifies the proofs, but also helps to make the methodology of analysis (simple application of Lyapunov theory) more transparent. With the assumption (12) given, the stability of (9) can be investigated by examining the stability of the system matrix

$$\begin{bmatrix} \underline{A} & \underline{B}\,\underline{\Lambda}\,\underline{G}_o^T \\ -\underline{H}_o\underline{C}^T & \underline{A} + \underline{B}\,\underline{G}_o^T + \underline{H}_o\underline{C}^T \end{bmatrix} \tag{13}$$

(in the sense that (13) is stable if all its eigenvalues have negative real parts).

Remark

In the above formulation we have assumed that the Kalman filter structure remains fixed at the nominal design values in the face of the control feedback perturbations. For greater generality, we can assume that some knowledge of the control perturbations may be 'communicated' to the Kalman filter design, or that the Kalman filter structure can be adjusted to 'track' the control perturbations in some manner to be specified. This can be incorporated into our problem formulation by

assuming that the filter structure is of the following form:

$$\begin{aligned}\dot{\hat{\underline{x}}}(t) &= (\underline{A} + \underline{B}\,\hat{\underline{\Lambda}}\,\underline{G}_o^T)\hat{\underline{x}}(t) + \underline{H}_o\underline{C}^T(\hat{\underline{x}}(t) - \underline{x}(t)) \\ &= (\underline{A} + \underline{B}\,\hat{\underline{\Lambda}}\,\underline{G}_o^T + \underline{H}_o\underline{C}^T)\hat{\underline{x}}(t) - \underline{H}_o\underline{C}^T\underline{x}(t)\end{aligned} \tag{14}$$

where $\underline{\Lambda}$ ≡ ('adjustable') filter structure parameter

= $\underline{I}$ nominally

By incorporating the assumptions in Remark 2 and 3, we therefore arrive at the following modified problem formulation:

LQG Stability Robustness Problem

For what $\underline{\Lambda} = \begin{bmatrix} \lambda_1 & & \underline{0} \\ & \ddots & \\ \underline{0} & & \lambda_m \end{bmatrix}$ and $\hat{\underline{\Lambda}} = \begin{bmatrix} \hat{\lambda}_1 & & 0 \\ & \ddots & \\ \underline{0} & & \hat{\lambda}_m \end{bmatrix}$

is the closed-loop system matrix

$$\begin{bmatrix} \underline{A} & \underline{B}\,\underline{\Lambda}\,\underline{G}_o^T \\ -\underline{H}_o\underline{C}^T & \underline{A} + \underline{B}\,\hat{\underline{\Lambda}}\,\underline{G}_o^T + \underline{H}_o\underline{C}^T \end{bmatrix} \tag{*}$$

stable?

Results

The main results we have obtained in the direction of sufficiency solutions to the LQG Stability Robustness Problem as formulated in the previous section will be presented in this section in the form of four

propositions. A fifth proposition generalizes the previous results to the more general time-varying gain-perturbation case. In arriving at these results we have utilized nothing more than simple applications of standard Lyapunov theory. The basic results from which all the Propositions in this section are derived has been stated as a Lemma (Lemma 4) in the Appendix.

Our first result pertains to the special case when we have perfect 'tracking' of the gain perturbation, i.e. when we have 'communicated' to the filter structure the exact perturbation values $\underline{\Lambda}$, so that $\hat{\underline{\Lambda}} \equiv \underline{\Lambda}$.

Proposition 1:

If $\hat{\underline{\Lambda}} \equiv \underline{\Lambda}$ then (*) is stable for all

$$\underline{\Lambda} > \frac{1}{2} (\underline{I} - \underline{X}_o^{-1}) \tag{15}$$

where

$$\underline{X}_o \triangleq (\underline{R}^{1/2} \underline{G}_o^T \underline{Q}^{-1} \underline{G}_o \underline{R}^{1/2})$$

i.e. The LQSF- gain margin is completely recovered. ■

Remark:

The condition $\hat{\underline{\Lambda}} \equiv \underline{\Lambda}$ in Proposition 1 ensures that the filter error dynamics are 'non-divergent'. This is best seen by examinig the error equation in detail:

$$\underline{e} = \hat{\underline{x}} - \underline{x}$$

$$\dot{\underline{e}} = (\underline{A} + \underline{H}_o\underline{C}^T + \underline{B}(\hat{\underline{\Lambda}} - \underline{\Lambda})G_o^T)\underline{e} + \underline{B}(\hat{\underline{\Lambda}} - \underline{\Lambda})\underline{G}_o^T\underline{x} \tag{16}$$

if $\hat{\underline{\Lambda}} \equiv \underline{\Lambda}$ then the 'feedback' driving term from the plant-state drops out, and the extra term in the system matrix of the error dynamics disappears.

Remark:

Since LQSF- gain margin may be wide enough to tolerate some channel failures (see [3]), Proposition 1 guarantees that such reliability of LQSF design remains with LQG provided corresponding change in the filter structure is made.

Proposition 1 assures us that full recovery of LQSF gain-margins is guaranteed with perfect knowledge of gain perturbations incorporated within the filter structure ('non-divergent' estimation, see [1],[5] for more details). There is another special situation under which similarly full recovery of LQSF gain margin can be guaranteed; this is the substance of our next proposition:

Proposition 2

If

$$\frac{\min\limits_{\underline{x}^T\underline{x}=1} \underline{x}^T(-(\underline{A}+\underline{H}_o\underline{C}^T)\underline{x}]}{\max\limits_{\underline{x}^T\underline{x}=1} \underline{x}^T(-(\underline{A}+\underline{B}\underline{G}_o^T)\underline{x}]} \to \infty \tag{17}$$

Then $(*)$ is stable for all

$$\underline{\Lambda} > \frac{1}{2}\,(\underline{I} - \underline{X}_o^{-1})$$

i.e. the LQSF-gain margin is recovered in the limit as the ratio (17) tends to infinity.

While Proposition 2 demonstrates that full LQSF-gain margins are recovered in a specific limit, it does not tell us anything about the 'rate of convergence' to gain-margin recovery, i.e. the explicit dependence of gain margins on the ratio (17). The following Proposition provides a partial answer to this question by giving an explicit sufficient bounds on the gain margins.

<u>Proposition 3</u>

Suppose $\hat{\underline{\Lambda}} \equiv \underline{I}$. Then $(*)$ is stable for all $\underline{\Lambda} > \underline{0}$ s.t.

$$\lambda_- \underline{I} < \underline{\Lambda} < \lambda_+ \underline{I} \tag{18}$$

where

$$\lambda_+ \triangleq 1 + 1/\sqrt{\omega_o} \tag{19}$$

$$\lambda_- \triangleq \max\{1 - (1/x_o),\ 1/\lambda_+\} \tag{20}$$

with

$$\omega_o \triangleq \max \lambda(\underline{W}_o) \tag{21a}$$

$$\underline{W}_o \triangleq \underline{R}^{-1/2}\underline{B}^T\underline{K}_f\underline{Q}^{-1}\underline{K}_f\underline{B}\underline{R}^{-1/2} \tag{21b}$$

$$\underline{K}_f(\underline{A} + \underline{H}_o\underline{C}^T) + (\underline{A} + \underline{H}_o\underline{C}^T)^T\,\underline{K}_f + \underline{Q} + \underline{G}_o\underline{R}\underline{G}_o^T = \underline{0} \tag{21c}$$

$$x_o \triangleq \max \lambda(\underline{X}_o) \tag{21d}$$

$$\underline{X}_o \triangleq \underline{R}^{1/2}\,\underline{G}_o^T\,\underline{Q}^{-1}\underline{G}_o\underline{R}^{1/2} \tag{21e}$$

Moreover, ω_o satisfies the following bound:

$$\sqrt{\omega_o} \leq \lambda_{min}(\underline{K}) \, \frac{\lambda_{max}(\underline{Q}+\underline{G}_o\underline{R}\underline{G}_o^T)}{\lambda_{min}(\underline{Q}+\underline{G}_o\underline{R}\underline{G}_o^T)} \cdot \frac{\max\limits_{\underline{x}^T\underline{x}=1} [\underline{x}^T(-(\underline{A}+\underline{B}\underline{G}_o^T)\underline{x}]}{\min\limits_{\underline{x}^T\underline{x}=1} [\underline{x}^T(-(\underline{A}+\underline{H}_o\underline{C}^T)\underline{x}]} \cdot \sqrt{\frac{\lambda_{max}(\underline{R}^{-1}\underline{B}^T\underline{B})}{\lambda_{min}(\underline{Q})}} \tag{22}$$

∎

Remark

The bounds on $\underline{\Lambda}$ obtained in Proposition 3 are only sufficient, and are in general rather conservative. Moreover, they are not the tightest possible bounds that can be derived from our approach (they have been essentially derived from Lemma 4 by choosing the parameter α to simplify the solution of the bounds rather than to optimize them, which entail much more tedious algebraic manipulations). The actual numerical computation of the parameter ω_o is straightforward although a bit tedious (requiring the solution of a Lyapunov equation (21)), but perhaps of greater theoretical significance is the simple bound on ω_o given in Equation (22). Taken together, Equation (18) and Equation (22) show that the upper bound on the gain-margin increment increases at least linearly with the ratio

$$\frac{\min\limits_{\underline{x}^T\underline{x}=1} [\underline{x}^T(-(\underline{A}+\underline{H}_o\underline{C}^T)\underline{x}]}{\max\limits_{\underline{x}^T\underline{x}=1} [\underline{x}^T(-(\underline{A}+\underline{B}\underline{G}_o^T)\underline{x}]}$$

Remark

In the results presented so far, no use has actually been made of the fact that the filter incorporated in the feedback loop is a Kalman filter; the only information of the filter we have utilized is the filter error dynamics matrix $(\underline{A} + \underline{H}_o\underline{C}^T)$ which could well have been designed by any other methods. More generally, therefore, the above results actually apply to any full-state filter design incorporated in the control feedback loop, and we can conjecture that similar versions of Proposition 2 and 3 apply in the case of any estimator dynamic compensator incorporated in the feedback loop.

Remark: Note that $\underline{Q} > \underline{0}$ is crucial for Proposition 3 to hold.

The next proposition, unlike the previous ones, explicitly make use of the assumption that the filter design incorporated in the control feedback loop is a Kalman filter. The basic question of interest is, for what choice of the noise specification $\underline{\Phi}$, $\underline{\Theta}$ will it be possible for the LQSF-gain margins to be fully recovered? Proposition 4 provides one answer.

Proposition 4

Let $\underline{\hat{\Lambda}} \equiv \underline{I}$ and suppose that $\underline{\Phi} = \underline{\Phi}_1 + \phi_2 \underline{B}\ \underline{\Xi}\ \underline{B}^T$ for some $\underline{\Xi} > \underline{0}$ and scalar $\phi_2 > 0$, and where $\underline{\Phi}_1 > 0$,

Then as $\phi_2 \to \infty$

The gain margins of $\underline{\Lambda}$ approaches

$$\underline{\Lambda} > \frac{1}{2}\,(\underline{I} - \underline{X}_o^{-1}) \qquad \blacksquare$$

i.e. full LQSF-gain margins are recovered as the plant driving noise-terms entering directly through the control input channels becomes greatly dominant (i.e. if $\underline{B}\ \underline{u}^*$ is the control term, the noise term $\underline{w}_2$ that gives rise to the variance $\phi_2\ \underline{B}\ \underline{\Xi}\ \underline{B}^T$ enters as $\underline{B}(\underline{u}^* + \underline{w}_2)$).

Remark

Proposition 4 is essentially the same as Doyle's result in Doyle [6] (where his assumptions are slightly different, and unnecessary, from ours) but our proof technique is completely different from his (which is a 'frequency domain' computation) and moreover our initial motivation has been independent from his work.

The most natural interpretation of Proposition 4 is that, for those systems <u>whose plant driving noise-terms enter primarily through the same channel as the control inputs</u> (hence the form of the noise-variance term $\phi_2 \underline{B}\ \underline{\Xi}\ \underline{B}^T$) themselves, recovery of LQSF-gain margins tends to be facilitated, with recoverry complete if these plant driving-noise terms

becomes greatly dominant over the observation channel noise. This makes sense intuitively, as Doyle pointed out, because the noise that enters the plant through the same channels as the control inputs can be interpreted as perturbations on the control inputs themselves, and this will get communicated through the mathematics into the filter design in such a way as to provide 'hedges' for the uncertainties in the control inputs.

Remark

Although we have assumed $\hat{\underline{\Lambda}} \equiv \underline{I}$ in in Proposition 4 (as this is the case of interest), this assumptions is actually not necessary - any finite $\hat{\underline{\Lambda}}$ will do, as is obvious from the proof. Of course, this can only be true in the case as $\phi_2 \to \infty$. For large but finite ϕ_2, there can be great differences on the gain-margins depending on what value $\hat{\underline{\Lambda}}$ takes.

Example 1

To illustrate the above propositions, consider the following single state, single control and single output system:

$$\underline{A} = a > 0, \quad \underline{B} = b \quad \underline{C}^T = c$$

$$\underline{Q} = q, \qquad \underline{R} = 1$$

$$\underline{\Phi} = \phi, \qquad \underline{\Theta} = 1$$

The regulator design is:

$$k = \frac{a}{b^2}\left[1 + \sqrt{1 + (1/\eta)}\right] \qquad \text{(Riccati matrix)}$$

$$g_o = -\frac{a}{b}\left[1 + \sqrt{1 + (1/\eta)}\right] \qquad \text{(Optimal gains)}$$

$$x_o = \left[\sqrt{\eta} + \sqrt{1+\eta}\,\right]^2 \qquad (\underset{\sim}{R}^{1/2}\ \underset{\sim o}{G}^T \underset{\sim}{Q}^{-1}\ \underset{\sim o}{G}\ \underset{\sim}{R}^{1/2})$$

$$a + bg_o = -\left(\sqrt{1 + 1/\eta}\,\right)a \qquad \text{(closed loop dynamics)},$$

where $\eta \triangleq a^2/qb^2$. Also, by duality, the filter design is

$$\sigma = \frac{a}{c^2}\ [1 + \sqrt{1 + 1/\eta_f}\,] \qquad \text{(covariance matrix)}$$

$$h_o = -\frac{a}{c}\ [1 + \sqrt{1 + 1/\eta_f}\,] \qquad \text{(filter gains)}$$

$$a + h_o c = -\left(\sqrt{1 + 1/\eta_f}\,\right)a \qquad \text{(error dynamics)}$$

where $\eta_f \triangleq a^2/\phi c^2$. Then the K_f matrix of equation (21) is given by

$$K_f = \frac{q}{2a\sqrt{1 + \dfrac{\phi c^2}{a^2}}} \left\{1 + \left[\sqrt{\eta} + \sqrt{1+\eta}\,\right]^2\right\},$$

and ω_o of (19) is

$$\sqrt{\omega_o} = \frac{\sqrt{qb}}{2a\sqrt{1 + \dfrac{\phi c^2}{a^2}}}\ (1 + x_o)$$

$$= \left(\sqrt{\frac{q}{\phi}}\right)\left(\frac{b}{c}\right)\left(\frac{1+x_o}{2}\right)\frac{1}{\sqrt{1+\eta_f}}$$

If we let q, ϕ to be such that

$$q \gg \left(\frac{a}{b}\right)^2 \qquad \text{(i.e. a wide band regulator)}$$

$$\phi \gg \left(\frac{a}{c}\right)^2 \qquad \text{(i.e. a wide band regulator)}$$

Then

$$\eta = \frac{1}{q}\left(\frac{a}{b}\right)^2 \to 0$$

$$\eta_f = \frac{1}{\phi}\left(\frac{a}{c}\right)^2 \to 0$$

and $x_o \to 1$

so

$$\omega_o \to \sqrt{\frac{qb^2}{\phi c^2}}$$

and we have

$$\max\{0, \lambda_+\} < \lambda < 1 + \sqrt{\frac{\phi c^2}{qb^2}}$$

or

$$\frac{1}{1 + \sqrt{\frac{\phi c^2}{qb^2}}} < \lambda < 1 + \sqrt{\frac{\phi c^2}{qb^2}}$$

The following representative values of λ_+ and λ_- as a function of $\left(\frac{\phi c^2}{qb^2}\right)$ are illustrative:

$\left(\frac{\phi c^2}{qb^2}\right) =$	<<1	1/100	1/16	1/4	1	4	16	100	>>1
λ_+	~1	11/10	5/4	3/2	2	3	5	11	∞
λ_-	~1	10/11	4/5	2/3	1/2	1/3	1/5	1/11	0

The results presented so far have been restricted to the case of control perturbations of the form:

$$\underline{u}^*(t) \longmapsto \underline{\Lambda}\, \underline{u}^*(t) \qquad \text{for all } t \in [0,\infty) \tag{24}$$

where $\underline{\Lambda}$ is a constant matrix. We now consider the more general case when the perturbations are given as time-varying:

$$\underline{u}^*(t) \longmapsto \underline{\Lambda}(t)\underline{u}^*(t) \qquad t \in [0,\infty) \tag{25}$$

As it turns out, the extension of our results in the previous section to the more general case (25) is trivially simple:

<u>Proposition 5</u>

If for each $t \in [0,\infty)$ point-wise, the functions $\underline{\Lambda}(t)$ and $\hat{\underline{\Lambda}}(t)$ satisfies the conditions on $\underline{\Lambda}$ and $\hat{\underline{\Lambda}}$ in any of the previous Propositions, then that proposition holds for the time-varying perturbations $\{\underline{\Lambda}(t),\ t \in [0,\infty)\}$, and $\{\hat{\underline{\Lambda}}(t),\ t \in [0,\infty)\}$.

■

Remark

Recall that, in general, if $\{\underline{\tilde{A}}(t),\ t \in [t_1,t_2]\}$ is such that for each $t \in [t_1,t_2]$ pointwise, $\underline{\tilde{A}}(t)$ has all its eigenvalues with negative real parts, it still need NOT be true that

$$\dot{\underline{\tilde{x}}}(t) = \underline{\tilde{A}}(t)\underline{x}(t)$$

is stable (in the sense that $\underline{\tilde{x}}(t) \to 0$ as $t \to \infty$). Thus one cannot 'prove' Proposition 5 by arguing that, if

$$\underline{\tilde{A}}(t) \triangleq \begin{bmatrix} \underline{A} & \underline{B}\,\underline{\Lambda}(t)\underline{G}_o^T \\ -\underline{H}_o\underline{C}^T & \underline{A} + \underline{B}\,\hat{\underline{\Lambda}}(t)\underline{G}_o^T + \underline{H}_o\,\underline{C}^T \end{bmatrix} \text{ is a stable matrix}$$

(all eigenvalues have negative real parts) for each $t \in [0,\infty)$ pointwise, then

$$\dot{\underline{\tilde{x}}}(t) = \underline{\tilde{A}}(t)\underline{\tilde{x}}(t), \qquad t \in [0,\infty)$$

is stable ($\underline{\tilde{x}}(t) \to 0$ as $t \to \infty$). The fact that Proposition 5 nevertheless does hold is because of the guaranteed existence of a single Lyapunov matrix for all $t \in [0,\infty)$.

Remark

The perturbation class $\{\underline{\Lambda}(t),\ t \in [0,\infty)\}$ can be trivially extended to the more general one of $\{\underline{\Lambda}(\{\underline{x}(\tau),\ \underline{u}^*(\tau),\ \tau \in [0,t]\},t),\ t \in [0,\infty)\}$ which incorporates dependence on $\underline{x}(\cdot)$ and $\underline{u}^*(\cdot)$.

Future Research Directions

Several areas of potentially fruitful research are readily suggested by the preliminary results we have obtained so far. We shall briefly list some of these below, not necessarily in any order of suggested priority.

1. Determining the 'rate of convergence' to LQSF-gain margin recovery by the noise specification

   $$\Phi = \Phi_1 + \phi_2 B \Xi B^T \quad \text{as } \phi_2 \text{ varies}$$

   Although Proposition 4 [and Doyle] has suggested the desirability of using noise specification of the above form for gain-margin consideration, that result, like Proposition 2, is a limit characterization that provides no clues as to the behavior of the gain-margins as ϕ_2 varies. An explicit sufficiency bounding solution similar in form to that of Proposition 3 which can demonstrate the dependence of some sufficient gain-margins bounding on ϕ_2 will be highly useful in practical design. Since the proof of Proposition 4 uses a procedure closely similar to that of Proposition 2 and 3, it appears that such a sufficiency bounding can be similarly derived for Proposition 4. We have not had sufficient time to investigate this.

2. As noted, the sufficiency bounds in Proposition 3 are not the tightest possible that can be derived from Lemma 4. A more careful effort in optimizing the bounds by exhausting all the free variables provided by Lemma 4 may lead to significantly tighter bounds.

3. Proposition 2 and 3 have pointed out that, purely from the gain-margin maximization point of view, those filters that have large ratios

$$\frac{\min[-\underline{x}^T(\underline{A}+\underline{H}_o\underline{C}^T)\underline{x}]}{\max[-\underline{x}^T(\underline{A}+\underline{B}\underline{G}_o^T)\underline{x}]} \qquad \text{(P)}$$

tend to be better. The following question therefore arises: Is there a simple way to classify the set of all possible design parameters

$$(\underline{Q},\underline{R},\underline{\Phi},\underline{\Theta})$$

into those combinations that have the property (P) and those that tend to be otherwise?

4. We have introduced the parameter $\underline{\hat{\Lambda}}$ into our formulation as it provides a natural interpretation of how a filter might adapts its structure to minimize 'divergence'. The actual implementational consideration of this 'gain-perturbation tracking' concept may lead to practical design significance (e.g. how to modify filter structure when there is control channel failure to guarantee stability of LQG system. The parameter $\underline{\hat{\Lambda}}$ tells us what needs to be changed).

5. Extension to discrete-time system, similar to what has been done for the LQSF case [5].

6. Application of results (especially Proposition 3 and 4) to some real physical systems (e.g. aircraft) to study the actual behavior of gain-margin bounds as ϕ_2 or $(\underline{A}+\underline{H}_o\underline{C}^T)$ are varied.

APPENDIX

The following lemmas will be useful in the proofs of the propositions in this paper.

Lemma 1

If $\underline{M}_1 \in R^{nxn}$, $\underline{M}_2 \in R^{mxm}$ are symmetric and $\underline{H} \in R^{nxm}$ is arbitrary, then

$$\left.\begin{aligned} &\underline{M}_1 > \underline{0} \\ &(\underline{M}_1 + \underline{M}_1\underline{H}\underline{M}_2\underline{H}^T\underline{M}_1) > \underline{0}\Big|_{R(\underline{H})} \end{aligned}\right\} \Rightarrow \underline{M}_1 + \underline{M}_1\underline{H}\underline{M}_2\underline{H}^T\underline{M}_1 > \underline{0}$$

Proof

See [2], Lemma 1 proof.

Lemma 2

If $\underline{Q}_1 > \underline{0}$ and $\underline{Q}_2 > \underline{Q}_{21}\underline{Q}_1^{-1}\underline{Q}_{12}$

Then

$$\begin{bmatrix} \underline{Q}_1 & \underline{Q}_{12} \\ \underline{Q}_{21} & \underline{Q}_2 \end{bmatrix} > \underline{0}$$

Proof

See any standard Linear Algebra text.

Lemma 3

$$\underline{H}^T(\underline{M}_1 + \underline{H}\underline{M}_2\underline{H}^T)^{-1}\underline{H} = [\underline{M}_2 + (\underline{H}^T\underline{M}_1^{-1}\underline{H})^{-1}]^{-1}$$

where $\underline{M}_2 > \underline{0}$ and the inverses defined in the equation exist. ■

Proof

$$\underline{H}^T(\underline{M}_1 + \underline{H}\underline{M}_2\underline{H}^T)^{-1}\underline{H} = \underline{H}^T\{\underline{M}_1^{-1} - \underline{M}_1^{-1}\underline{H}(\underline{M}_2^{-1} + \underline{H}^T\underline{M}_1^{-1}\underline{H})^{-1}\underline{H}^T\underline{M}_1^{-1}\}\underline{H}$$

$$= (\underline{H}^T\underline{M}_1^{-1}\underline{H}) - (\underline{H}^T\underline{M}_1^{-1}\underline{H})(\underline{M}_2^{-1} + \underline{H}^T\underline{M}_1^{-1}\underline{H})^{-1}(\underline{H}^T\underline{M}_1^{-1}\underline{H})$$

$$= [\underline{M}_2 + (\underline{H}^T\underline{M}_1^{-1}\underline{H})^{-1}]^{-1}$$

■

Lemma 4

(*) is stable for $\underline{R} > \underline{0}$, $\underline{\Lambda}$ and $\hat{\underline{\Lambda}}$ diagonal if there exists a matrix $\underline{L} > \underline{0}$ and a matrix $\hat{\underline{Q}} > 0$ for which the following conditions hold:

(i) $2\underline{\Lambda} - (\underline{I} + \underline{L}) + X_o^{-1} > \underline{0} \iff \underline{\Lambda} > \frac{1}{2}((\underline{I}+\underline{L}) - X_o^{-1})$

(ii) $\hat{\underline{Q}} > \underline{G}_o\underline{R}^{1/2}[\underline{I} + \underline{X}_o^{-1}]^{-1}\underline{R}^{1/2}\underline{G}_o^T$

(iii) $\hat{\underline{Q}} > \underline{G}_o\{\underline{LR} + (\underline{\Lambda} - \underline{L})\underline{R}^{1/2}[2\underline{\Lambda} - (\underline{I}+\underline{L}) + \underline{X}_o^{-1}]^{-1} R^{1/2}(\underline{\Lambda}-\underline{L})\}G_o^T$

$\qquad + \hat{\underline{K}}\underline{B}(\underline{\Lambda}-\hat{\underline{\Lambda}})^2(\underline{L}\,\underline{R})^{-1}B^T\hat{K}$

where

$$\hat{\underline{K}}(\underline{A} + \underline{H}_o\underline{C}^T) + (\underline{A} + \underline{H}_o\underline{C}^T)^T\hat{\underline{K}} + \hat{\underline{Q}} = 0$$

■

Proof of Lemma 4

We have

$$\begin{bmatrix} \dot{\underline{x}} \\ \dot{\hat{\underline{x}}} \end{bmatrix} = \begin{bmatrix} \underline{A} & \underline{B}\underline{\Lambda}\underline{G}_o^T \\ -\underline{H}_o\underline{C}^T & \underline{A} + \underline{H}_o\underline{C}^T - \underline{B}\hat{\underline{\Lambda}}\underline{G}_o^T \end{bmatrix} \begin{bmatrix} x \\ \hat{x} \end{bmatrix} \qquad (A1)$$

Let $\underline{e} \equiv \hat{\underline{x}} - \underline{x}$

Then (A1) <=> $$\begin{bmatrix} \dot{x} \\ \dot{e} \end{bmatrix} = \underbrace{\begin{bmatrix} A + B\Lambda G_o^T & B\Lambda G_o^T \\ B(\hat{\Lambda}-\Lambda)G_o^T & A + H_oC^T + B(\hat{\Lambda}-\Lambda)G_o^T \end{bmatrix}}_{\tilde{A}} \begin{bmatrix} x \\ e \end{bmatrix} \tag{A3}$$

The matrix $\tilde{A}$ can be rewritten as

$$\tilde{A} = \underbrace{\begin{bmatrix} A + BG_o^T & BG_o^T \\ 0 & A + H_oC^T \end{bmatrix}}_{\tilde{A}_o} + \underbrace{\begin{bmatrix} B(\Lambda-I)G_o^T & B(\Lambda-I)G_o^T \\ B(\hat{\Lambda}-\Lambda)G_o^T & B(\hat{\Lambda}-\Lambda)G_o^T \end{bmatrix}}_{\delta\tilde{A}} \tag{A4}$$

Since $\tilde{A}_o$ is a stable matrix, for every $\tilde{Q} > 0$ there exists a $\tilde{K} > 0$ such that

$$\tilde{K}\tilde{A}_o + \tilde{A}_o^T\tilde{K} + \tilde{Q} = 0 \tag{A5}$$

by Lyapunov Theorem.

If we choose $$\tilde{Q} \triangleq \begin{bmatrix} Q + G_oRG_o^T & G_oRG_o^T \\ G_oRG_o^T & \hat{Q} \end{bmatrix} > 0 \tag{A6}$$

(where $\hat{Q} > 0$ is to be specified s.t. (A6) holds)

Then (A5) => $$\tilde{K} = \begin{bmatrix} K & 0 \\ 0 & \hat{K} \end{bmatrix} \tag{A7}$$

where $\underline{K} > \underline{0}$ is the unique solution to the Riccati equation:

$$\underline{KA} + \underline{A}^T\underline{K} - \underline{KBR}^{-1}\underline{B}^T\underline{K} + \underline{Q} = \underline{0} \tag{A8}$$

and $\hat{\underline{K}} > \underline{0}$ is the unique solution to the Lyapunov equation:

$$\hat{\underline{K}}(\underline{A} + \underline{H}_o\underline{C}^T) + (\underline{A} + \underline{H}_o\underline{C}^T)^T\,\hat{\underline{K}} + \hat{\underline{Q}} = \underline{0} \tag{A9}$$

Further, from (A5), we have

$$\tilde{\underline{K}}(\tilde{\underline{A}}_o + \delta\tilde{\underline{A}}) + (\tilde{\underline{A}}_o + \delta\tilde{\underline{A}})^T\,\tilde{\underline{K}} + \delta\tilde{\underline{Q}} = \underline{0} \tag{A10}$$

where

$$\delta\tilde{\underline{Q}} \triangleq \tilde{\underline{Q}} - (\tilde{\underline{K}}\delta\tilde{\underline{A}} + \delta\tilde{\underline{A}}^T\tilde{\underline{K}}) \tag{A11}$$

so from Lyapunov theorem, we know that $(\underline{A}_o + \delta\tilde{\underline{A}})$ is stable if

$$\delta\tilde{\underline{Q}} > \underline{0} \tag{A12}$$

Given the choice of $\tilde{\underline{Q}}$ as in (A6) and the corresponding form of $\tilde{\underline{K}}$ as in (A7), we get (with $\underline{R}$ diagonal)

$$\delta\tilde{\underline{Q}} = \begin{bmatrix} \underline{Q} + \underline{G}_o(2\underline{\Lambda}-\underline{I})\underline{RG}_o^T & \underline{G}_o\underline{R\Lambda G}_o^T + \underline{G}_o(\underline{\Lambda}-\hat{\underline{\Lambda}})\underline{B}^T\hat{\underline{K}} \\ \underline{G}_o\underline{R\Lambda G}_o^T + \hat{\underline{K}}\underline{B}(\underline{\Lambda}-\hat{\underline{\Lambda}})\underline{G}_o^T & \hat{\underline{Q}} + \underline{G}_o(\underline{\Lambda}-\hat{\underline{\Lambda}})\underline{B}^T\hat{\underline{K}} + \hat{\underline{K}}\underline{B}(\underline{\Lambda}-\hat{\underline{\Lambda}})\underline{G}_o^T \end{bmatrix} \tag{A13}$$

From Lemma 2, the condition (A6) is equivalent to:

$$\begin{cases} \underline{Q} + \underline{G}_o\underline{RG}_o^T > \underline{0} & \text{(which is true from definition of } \underline{Q}) \\ \hat{\underline{Q}} > \underline{G}_o\underline{RG}_o^T(\underline{Q} + \underline{G}_o\underline{RG}_o^T)^{-1}\underline{G}_o\underline{RG}_o^T & \end{cases} \tag{A14}$$

Since (A14) <=> $\hat{\underline{Q}} > \underline{G}_o\underline{R}^{1/2}(\underline{I} + \underline{X}_o^{-1})^{-1}\ \underline{R}^{1/2}\underline{G}_o^T$ (from Lemma 3)

where $\underline{X}_o \triangleq (\underline{R}^{1/2}\underline{G}_o^T\underline{Q}^{-1}\underline{G}_o\underline{R}^{1/2})$ (A16)

what remains to be proved is that the conditions in Lemma 4 are sufficient for $\delta\tilde{\underline{Q}} > \underline{0}$. Now, from (A13), we have for any diagonal $\underline{L}$

$$\delta\tilde{\underline{Q}} = \begin{bmatrix} \tilde{\underline{Q}} + \underline{G}_o(2\underline{\Lambda}-(\underline{I}+\underline{L})\underline{R}\underline{G}_o^T & \underline{G}_o\underline{R}(\underline{\Lambda}-\underline{L})\underline{G}_o^T \\ \underline{G}_o\underline{R}(\Lambda-\underline{L})\underline{G}_o^T & \hat{\underline{Q}} - \underline{G}_o\underline{R}\underline{L}\underline{G}_o^T - \hat{\underline{K}}\underline{B}(\Lambda-\hat{\Lambda})^2(\underline{L}\underline{R})^{-1}\underline{B}^T\hat{\underline{K}} \end{bmatrix}$$

$$+\begin{bmatrix} \underline{G}_o \\ \underline{G}_o + \hat{\underline{K}}\underline{B}(\underline{\Lambda}-\hat{\underline{\Lambda}})\underline{R}^{-1}\underline{L}^{-1} \end{bmatrix} \underline{L}\underline{R}[\underline{G}_o^T,\ \underline{G}_o^T + \underline{L}^{-1}\underline{R}^{-1}(\underline{\Lambda}-\hat{\underline{\Lambda}})\underline{B}^T\hat{\underline{K}}] \quad \text{(A17)}$$

If $\underline{L} > 0$, then the second term of (A17) is positive semidefinite, so if the first term of (A17) is > 0, then it will follow that $\delta\tilde{\underline{Q}} > 0$. But sufficient conditions for the first term of (A17) to be $> \underline{0}$ are as follows (from Lemma 2):

$$\begin{cases} \underline{Q} + \underline{G}_o(2\underline{\Lambda}-(\underline{I}+\underline{L}))\ \underline{R}\underline{G}_o^T > \underline{0} & \text{(A18)} \\ \hat{\underline{Q}} > \underline{G}_o(\underline{\Lambda}-\underline{L})\underline{R}\underline{G}_o^T[\underline{Q} + \underline{G}_o(2\Lambda-(\underline{I}+\underline{L})\ \ \underline{R}\underline{G}_o^T]^{-1}\ \underline{G}_o\underline{R}(\underline{\Lambda}-\underline{L})\underline{G}_o^T & \text{(A19)} \end{cases}$$

$$+\ \underline{G}_o\underline{L}\underline{R}\underline{G}_o^T + \hat{\underline{K}}\underline{B}(\underline{\Lambda}-\hat{\underline{\Lambda}})^2(\underline{L}\underline{R})^{-1}\underline{B}^T\hat{K}$$

and (A18) <=> $\underline{\Lambda} > \frac{1}{2}((\underline{I}+\underline{L}) - \underline{X}_o^{-1})$ (after applying Lemma 1) (A20)

while (A19) <=>

$$\hat{\underline{Q}} > \underline{G}_o\{\underline{L}\underline{R} + (\underline{\Lambda}-\underline{L})\underline{R}^{1/2}[2\underline{\Lambda}-(\underline{I}+\underline{L}) + \underline{X}_o^{-1}]^{-1}\ \underline{R}^{1/2}(\underline{\Lambda}-\underline{L})\}\underline{G}_o^T$$

$$+\ \hat{\underline{K}}\underline{B}(\underline{\Lambda}-\hat{\underline{\Lambda}})^2(\underline{L}\underline{R})^{-1}\underline{B}^T\hat{\underline{K}} \quad \text{(A21)}$$

(after applying Lemma 3)

■

Lemma 5

Let $\tilde{A}$ be a stable matrix, $\tilde{Q} > 0$, and $\tilde{K} > 0$ be the unique solution to the Lyapunov equation

$$\tilde{K}\tilde{A} + \tilde{A}^T\tilde{K} + \tilde{Q} = 0$$

then

$$\lambda_{max}(\tilde{K}) \le \frac{\lambda_{max}(\tilde{Q})}{2 \min\limits_{\substack{x \neq o \\ x^T x = 1}} [x^T(-\tilde{A})x]}$$

$$\lambda_{min}(\tilde{K}) \ge \frac{\lambda_{min}(\tilde{Q})}{2 \max\limits_{\substack{x \neq o \\ x^T x = 1}} [x^T(-\tilde{A})x]}.$$

■

Proof

We have

$$\tilde{K}(-\tilde{A}) + (-\tilde{A})^T\tilde{K} = \tilde{Q}$$

$$x^T(\tilde{K}(-\tilde{A}) + (-\tilde{A})^T\tilde{K})x = x^T\tilde{Q}x \qquad \forall\, x \neq 0, \quad x^T x = 1$$

$$x^T(\tilde{K}(-\tilde{A}) + (-\tilde{A})^T\tilde{K})x \le \lambda_{max}(Q) \qquad \forall\, x \neq 0, \quad x^T x = 1$$

Now chose $x = \tilde{x} \neq 0$ such that

$$\tilde{K}\tilde{x} = \lambda_{max}(\tilde{K})\, \tilde{x}, \qquad \tilde{x}^T\tilde{x} = 1$$

Then

$$2\lambda_{max}(\underline{\tilde{K}})\,[\underline{\tilde{x}}^T(-\underline{\tilde{A}})\underline{\tilde{x}}] \leq \lambda_{max}(\underline{\tilde{Q}})$$

Now

$$\min_{\substack{\underline{x}\neq 0 \\ \underline{x}^T\underline{x}=1}} \underline{x}^T(-\underline{\tilde{A}})\underline{x} \leq \underline{\tilde{x}}^T(-\underline{\tilde{A}})\underline{\tilde{x}}$$

so

$$2\lambda_{max}(\underline{\tilde{K}}) \left\{ \min_{\substack{\underline{x} \\ \underline{x}\neq 0 \\ \underline{x}^T\underline{x}=1}} [\underline{x}^T(-\underline{\tilde{A}})\underline{x}] \right\} \leq \lambda_{max}(\underline{\tilde{Q}})$$

or

$$\lambda_{max}(\underline{\tilde{K}}) \leq \frac{\lambda_{max}(\underline{\tilde{Q}})}{2 \min\limits_{\substack{\underline{x}\neq 0 \\ \underline{x}^T\underline{x}=1}} [\underline{x}^T(-\underline{\tilde{A}})\underline{x}]}$$

Similarly, we can write

$$\underline{x}^T(\underline{\tilde{K}}(-\underline{\tilde{A}}) + (-\underline{\tilde{A}})^T\underline{\tilde{K}})\underline{x} \geq \lambda_{min}(\underline{\tilde{Q}}) \qquad \begin{matrix}\underline{x}\neq 0 \\ \underline{x}^T\underline{x}=1\end{matrix}$$

and choose $\underline{x}^*\neq 0$, $\underline{x}^{*T}\underline{x}^*=1$ s.t.

$$\underline{\tilde{K}}\underline{x}^* = \lambda_{min}(\underline{\tilde{K}})\underline{x}^*.$$

Then $2\lambda_{min}(\underline{\tilde{K}})\,[\underline{x}^{*T}(-\underline{\tilde{A}})\underline{x}^*] \geq \lambda_{min}(\underline{\tilde{Q}})$

and

$$\lambda_{min}(\underline{\tilde{K}}) \geq \frac{\lambda_{min}(\underline{\tilde{Q}})}{2 \max\limits_{\substack{\underline{x}\neq 0 \\ \underline{x}^T\underline{x}=1}} [\underline{x}^T(-\underline{\tilde{A}})\underline{x}]} \quad .$$

■

Lemma 6

Let

$$\omega_o \triangleq \lambda_{max}(W_o) \triangleq \lambda_{max}(R^{-1/2}B^T K_f Q^{-1} K_f B R^{-1/2})$$

with

$$K_f(A + H_o C^T) + (A + H_o C^T)^T K_f + Q + G_o R G_o^T = 0$$

Then

$$\sqrt{\omega_o} \leq \lambda_{min}(K) \cdot \frac{\lambda_{max}(Q+G_o R G_o^T)}{\lambda_{min}(Q+G_o R G_o^T)} \cdot \frac{\max\limits_{x^T x=1} [x^T(-(A+BG_o^T)x]}{\min\limits_{x^T x=1} [x^T(-(A+H_o C^T)x]} \cdot \sqrt{\frac{\lambda_{max}(R^{-1}B^T B)}{\lambda_{min}(Q)}}$$

■

Proof of Lemma 6

We have

$$K_f[-(A+H_o C^T)] + [-(A+H_o C^T)]^T K_f = Q + G_o R G_o^T$$

$$K[-(A+BG_o^T)] + [-(A+BG_o^T)]^T K = Q + G_o R G_o^T$$

so from Lemma 5

$$\lambda_{max}(K_f) \leq \frac{\lambda_{max}(Q+G_o R G_o^T)}{2 \min\limits_{\substack{x \neq 0 \\ x^T x=1}} [x^T(-(A+H_o C^T))x]}$$

$$\lambda_{min}(K) \geq \frac{\lambda_{min}(Q+G_o R G_o^T)}{2 \max\limits_{\substack{x \neq 0 \\ x^T x=1}} [x^T(-(A+BG_o^T))x]}$$

and hence

$$\lambda_{max}(\underline{K}_f) \leq \frac{\lambda_{max}(\underline{Q}+\underline{G}_o\underline{R}\underline{G}_o^T)}{\lambda_{min}(\underline{Q}+\underline{G}_o\underline{R}\underline{G}_o^T)} \frac{\max\limits_{\underline{x}^T\underline{x}=1}\left[\underline{x}^T(-(\underline{A}+\underline{B}\underline{G}_o^T))\underline{x}\right]}{\min\limits_{\underline{x}^T\underline{x}=1}\left[\underline{x}^T(-(\underline{A}+\underline{H}_o\underline{C}^T))\underline{x}\right]} \lambda_{min}(\underline{K}) .$$

We also have

$$\begin{aligned}\omega_o &= \lambda_{max}(\underline{R}^{-1/2}\underline{B}^T\underline{K}_f\underline{Q}^{-1}\underline{K}_f\underline{B}\underline{R}^{-1/2}) \\ &\leq \lambda_{max}(\underline{K}_f\underline{Q}^{-1}\underline{K}_f)\lambda_{max}(\underline{B}\underline{R}^{-1}\underline{B}^T) \\ &\leq \lambda_{max}(\underline{K}_f^2)\lambda_{max}(\underline{Q}^{-1})\lambda_{max}(\underline{B}\underline{R}^{-1}\underline{B}^T) \\ &\leq [\lambda_{max}(\underline{K}_f)]^2[\lambda_{min}(\underline{Q})]^{-1}\lambda_{max}(\underline{R}^{-1}\underline{B}^T\underline{B})\end{aligned}$$

Hence,

$$\sqrt{\omega_o} \leq \lambda_{max}(\underline{K}_f)\sqrt{\frac{\lambda_{max}(\underline{R}^{-1}\underline{B}^T\underline{B})}{\lambda_{min}(\underline{Q})}} .$$

This and the bound on $\lambda_{max}(\underline{K}_f)$ establish the Lemma.

■

Proof of Proposition 1

If $\hat{\underline{\Lambda}} \equiv \underline{\Lambda}$ then by letting $\underline{L} = \alpha\underline{I}$, $\alpha > 0$ the conditions in Lemma 4 becomes

(i)' $\quad \underline{\Lambda} > \frac{1}{2}((1+\alpha)\underline{I} - \underline{X}_o^{-1})$

(ii)' $\quad \hat{\underline{Q}} > \underline{G}_o\underline{R}^{1/2}(\underline{I} + \underline{X}_o^{-1})^{-1}\underline{R}^{1/2}G_o$

(iii)' $\quad \hat{\underline{Q}} > \underline{G}_o\{\alpha\underline{R} + \underline{R}^{1/2}(\underline{\Lambda}-\alpha\underline{I})[2\underline{\Lambda}-(1+\alpha)\underline{I} + \underline{X}_o^{-1}]^{-1}(\underline{\Lambda}-\alpha\underline{I})\underline{R}^{1/2}\underline{G}_o^T$

By choosing $\hat{\underline{Q}}$ sufficiently large (positive definite) and by letting $\alpha \to 0$, conditions (ii)' & (iii)' are always satisfied while condition (i)' $\to \underline{\Lambda} > \frac{1}{2}(\underline{I} - \underline{X}_o^{-1})$.

■

Proof of Proposition 2

Let $\hat{Q} \triangleq \beta(Q + G_o R G_o^T)$ where $\beta > 0$ is chosen such that condition (ii) of Lemma 4 is satisfied, i.e.

$$\beta(Q + G_o R G_o^T) > G R^{1/2}[L + X_o^{-1}]^{-1} R^{1/2} G_o^T \quad . \tag{A23}$$

Then $\hat{K} = \beta K_f > 0$

where K_f is the unique Lyapunov solution of

$$K_f(A + H_o C^T) + (A + H_o C^T)^T K_f + Q + G_o R G_o^T = 0 \tag{A24}$$

Let $L = \alpha I$, $\alpha > 0$. Condition (iii) in Lemma 4 then becomes

$$\beta(Q - \frac{\beta}{\alpha} K_f B(\Lambda-\hat{\Lambda})^2 R^{-1} B^T K_f)$$
$$+ G_o\{(\beta-\alpha)R - R^{1/2}(\Lambda-\alpha I)[2\Lambda-(1+\alpha)I + X_o^{-1}]^{-1}(\Lambda-\alpha I)R^{1/2}\}G_o^T > 0 \tag{A25}$$

A sufficient conditions for (A25) to hold is therefore:

$$\begin{cases} Q > \frac{\beta}{\alpha} K_f B(\Lambda-\hat{\Lambda})^2 R^{-1} B^T K_f & \text{(A26)} \\ (\beta-\alpha)I - (\Lambda-\alpha I)[2\Lambda-(1+\alpha)I + X_o^{-1}]^{-1}(\Lambda-\alpha I) > 0 & \text{(A27)} \end{cases}$$

The condition $Q > 0$ and Lemma 1 can now be used to show that

(A26) <=>
$$(R^{-1/2} B^T K_f Q^{-1} K_f B R^{-1/2})^{-1} > \frac{\beta}{\alpha} (\Lambda-\hat{\Lambda})^2 \tag{A28}$$
or
$$(W_o)^{-1} > \frac{\beta}{\alpha} (\Lambda-\hat{\Lambda})^2 \tag{A29}$$

A sufficient condition for (A29) is

$$\frac{1}{\omega_o} I > \frac{\beta}{\alpha} (\Lambda-\hat{\Lambda})^2 \tag{A30}$$

where $\omega_o \overset{\Delta}{=} \lambda_{max}\ (\underset{\sim}{W}_o)$. If we now take

$$\beta = \sqrt[4]{\omega_o}\ ,\ \alpha = 1/\beta \tag{A31}$$

Then (A30) becomes

$$\frac{1}{\sqrt{\omega_o}}\ \underline{I}\ >\ (\underline{\Lambda}-\hat{\underline{\Lambda}})^2 \tag{A32}$$

But from Lemma 6,

$$\frac{1}{\sqrt{\omega_o}}\ \text{varies as}\ \frac{\underset{\underline{x}^T\underline{x}=1}{\min}\ [-\underline{x}^T(\underline{\Lambda}+\underline{H}_o\underline{C}^T)\underline{x}]}{\underset{\underline{x}^T\underline{x}=1}{\max}\ [-\underline{x}^T(\underline{\Lambda}+\underline{B}\underline{G}_o^T)\underline{x}]} \tag{A33}$$

so if this ratio tends to infinity then $(\underline{\Lambda}-\hat{\underline{\Lambda}})^2$ may become arbitrarily large.

Moreover, from (A31), $\alpha \to 0$ and $\beta \to \infty$, so the conditions (A23) and (A27) are satisfied for all $\underline{\Lambda}$ s.t. $[2\underline{\Lambda}-(1+\alpha)\underline{I} + \underline{X}_o^{-1}]^{-1}$ is finite. The only remaining condition of Lemma 4 is Condition (i), which tends to:

$$\underline{\Lambda} > \frac{1}{2}\ ((1+\alpha)\underline{I} - \underline{X}_o^{-1}) \longrightarrow \underline{\Lambda} > \frac{1}{2}\ (\underline{I} - \underline{X}_o^{-1})$$

■

Proof of Proposition 3

From the proof of Proposition 2, the following conditions are sufficient for (*) to be stable:

$$\begin{cases} \text{(a)} & \underline{\Lambda} > \frac{1}{2}\left((1+\alpha)\underline{I} - \underline{X}_o^{-1}\right) & \text{(A34)} \\ \text{(b)} & \underline{W}_o^{-1} > \frac{\beta}{\alpha}\,(\underline{\Lambda}-\hat{\underline{\Lambda}})^2 & \text{(A35)} \\ \text{(c)} & (\beta-\alpha)\underline{I} > (\underline{\Lambda}-\alpha\underline{I})\,[2\underline{\Lambda} - (1+\alpha)\underline{I} + \underline{X}_o^{-1}]^{-1}(\underline{\Lambda}-\alpha\underline{I}) & \text{(A36)} \\ \text{(d)} & \beta(\underline{Q} + \underline{G}_o\underline{R}\underline{G}_o^T) > \underline{G}_o\underline{R}^{1/2}[\underline{I} + \underline{X}_o^{-1}]^{-1}\,\underline{R}^{1/2}\,\underline{G}_o^T & \text{(A37)} \end{cases}$$

To simplify the proof consider first the case $\underline{\Lambda}=\lambda\underline{I}$, $\lambda > 0$, and $\hat{\underline{\Lambda}} = \underline{I}$. If we choose $\alpha = \lambda$, then conditions (a)-(c) become

$$\text{(a)}' \quad \lambda\,\underline{I} > (\underline{I} - \underline{X}_o^{-1}) \tag{A38}$$

$$\text{(b)}' \quad \underline{W}_o^{-1} > \frac{\beta(\lambda-1)^2}{\lambda}\,\underline{I} \tag{A39}$$

$$\text{(c)}' \quad \beta > \lambda \tag{A40}$$

Sufficient conditions for (a)'-(b)' to hold are:

$$\lambda > (1 - \frac{1}{x_o}) \qquad \text{where } x_o \overset{\Delta}{=} \max\lambda(X_o) \tag{A41}$$

$$\omega_o < \frac{\lambda}{\beta(\lambda-1)^2} \qquad \text{where } \omega_o \overset{\Delta}{=} \max\lambda(W_o) \tag{A42}$$

To find the upper bound λ_+ of given (A40),(A41),(A42) and (A37), set

$$\beta = \lambda_+ \tag{A43}$$

Then (A42) yields

$$\omega_o = \frac{1}{(\lambda_+ - 1)^2} \tag{A44}$$

or

$$\lambda_+ = 1 + \frac{1}{\sqrt{\omega_o}} > 1 \tag{A45}$$

We still need to check that (A37) is satisfied, i.e.

$$\lambda_+ (\underline{Q} + \underline{G}_o \underline{R} \underline{G}_o^T) > \underline{G}_o \underline{R}^{1/2} [\underline{I} + \underline{X}_o^{-1}]^{-1} \underline{R}^{1/2} \underline{G}_o^T \tag{A46}$$

It can be shown, abbeit with some amount of algebraic manipulations by applying Lemma 1 and 3, that a sufficient condition for (A46) is:

$$\lambda_+ > \frac{x_o^2}{(1+x_o)^2} \tag{A47}$$

Since $\left(\frac{x_o}{(1+x_o)}\right)^2 < 1$ and $\lambda_+ > 1$, (A47) is satisfied.

To find a lower bound λ_- of λ, we first find the lower bound λ_-' given by equation (A42). We have

$$\omega_o = \frac{\lambda_-'}{\lambda_+ (\lambda_-' - 1)^2} \tag{A48}$$

It can be shown, after some algebraic manipulation, that $\lambda_-' = 1/\lambda_+$ (A49).

Since λ must also satisfies the lower bound given by (A41), we therefore have

$$\lambda_- \triangleq \max \left\{ (1 - \frac{1}{x_o}),\ \lambda_-' \right\} \tag{A50}$$

We have thus shown that (*) is stable for all $\underline{\Lambda} = \lambda I$ (and $\hat{\underline{\Lambda}} \equiv I$) such that

$$\lambda_- < \lambda < \lambda_+ \tag{A51}$$

where λ_+ is defined by (A45) and λ_- by (A50). The generalization to the case

$$\lambda_- I < \underline{\Lambda} < \lambda_+ I \tag{A52}$$

for general diagonal $\underline{\Lambda}$ is obtained by replacing choice of $\underline{L} = \alpha \underline{I}$ by a general $\underline{L} > \underline{0}$ as provided by Lemma 4. We shall omit the details of the proof of this as it is straightforward (albeit tedious). ■

Proof of Proposition 4

Consider the Kalman filter Riccati equation:

$$\underline{\Sigma}(\underline{A} + \underline{H}_o \underline{C}^T)^T + (\underline{A} + \underline{H}_o \underline{C}^T)\underline{\Sigma} + \underline{\Sigma}\,\underline{C}\,\underline{\Theta}^{-1}\,\underline{C}^T \underline{\Sigma} + \Phi = \underline{0} \tag{A53}$$

<=>

$$(\underline{A} + \underline{H}_o \underline{C}^T)^T \underline{\Sigma}^{-1} + \underline{\Sigma}^{-1}(\underline{A} + \underline{H}_o \underline{C}^T) + \underline{C}\,\underline{\Theta}^{-1}\,\underline{C}^T + \underline{\Sigma}^{-1}\,\Phi\,\underline{\Sigma}^{-1} = \underline{0} \tag{A54}$$

If we let $\hat{\underline{Q}} \triangleq \beta(\underline{C}\underline{\Theta}^{-1}\underline{C}^T + \underline{\Sigma}^{-1}\,\underline{\Phi}\,\underline{\Sigma}^{-1})$ in Lemma 4 (such that Condition (ii) holds) then

$$\underline{K} = \beta \underline{K}^{-1} \tag{A55}$$

and Condition (iii) in Lemma 4 becomes

C-2

$$\beta((CQ)^{-1}C^{T} + \Sigma^{-1}\Phi\,\Sigma^{-1}) > G_{o}R^{1/2}\,MR^{1/2}G_{o}^{T} + \frac{\beta^{2}}{\alpha}\,\Sigma^{-1}\,B(\Lambda-\hat{\Lambda})^{2}\,R^{-1}B^{T}\Sigma^{-1} \quad \text{(A56)}$$

where, by letting $L = \alpha I$, $\alpha > 0$

$$M \triangleq \alpha I + (\Lambda-\alpha I)\,[2\Lambda-(1+\alpha)I + X_{o}^{-1}]^{-1}(\Lambda-\alpha I) \quad 0 \quad \text{(A57)}$$

If $\Phi \triangleq \Phi_{1} + \phi_{2}\,B\,\Xi\,B^{T}$ (A58)

Then (A56) $\Leftrightarrow$

$$\left.\begin{aligned}&[\beta(CQ^{-1}C^{T} + \Sigma^{-1}\,\Phi_{1}\,\Sigma^{-1})] - (G_{o}R^{1/2}\,MR^{1/2}G_{o}^{T}) + \\ &\beta\,\Sigma^{-1}\,B\left(\phi_{2}\,\Xi - \frac{\beta}{\alpha}\,(\Lambda-\hat{\Lambda})^{2}\,R^{-1}\right)B^{T}\,\Sigma^{-1} > 0 \quad (\Rightarrow \alpha \to o)\end{aligned}\right\} \quad \text{(A59)}$$

If we take $\alpha = 1/\beta$, and let $\phi_{2} \to \infty$, $\beta \to \infty$ in such a way that $\phi_{2}/\beta^{2} \to \infty$, then in the limit (A59) is satisfied for all $(\Lambda-\hat{\Lambda})^{2}$ and Λ finite and Condition (4.1) of Lemma (4) tends to: $\Lambda > \frac{1}{2}(I - X_{o}^{-1})$. ■

Proof of Proposition 5

By Lyapunov theorem, the dynamic system

$$\dot{\tilde{x}} = \tilde{A}(t)\tilde{x}(t), \quad t \in [0,\infty),\ \tilde{x}(0) \text{ given, finite} \quad \text{(A60)}$$

is stable ($\tilde{x}(t) \to 0$ as $t \to \infty$) if there exists a positive function s.t.

1) $v(\tilde{x}) > 0 \qquad \tilde{x} \neq 0$ (A61)

2) $\frac{dv}{dt}(\tilde{x}(t)) > 0 \qquad t \in [0,\infty)$ and $\tilde{x}(t)$ satisfying (A60) (A62)

If we now consider the function

$$\eta : \begin{bmatrix} \underline{x} \\ \underline{e} \end{bmatrix} = \underline{\tilde{x}} \longmapsto \underline{\tilde{x}}^T \underline{\tilde{K}} \underline{\tilde{x}} \quad \text{where } \underline{\tilde{K}} \text{ is the Lyapunov matrix} \tag{A63}$$

as defined in (A5).

then

(1) $\eta > 0 \qquad \underline{\tilde{x}} \neq 0 \qquad$ since $\underline{\tilde{K}} > \underline{0}$

and

(2) $$\dot{\eta} = \underline{\dot{\tilde{x}}}^T \underline{\tilde{K}} \underline{\tilde{x}} + \underline{\tilde{x}}^T \underline{\tilde{K}} \underline{\dot{\tilde{x}}} \tag{A64}$$

$$= \underline{\tilde{x}}^T (\underline{\tilde{A}}^T \underline{\tilde{K}} + \underline{\tilde{K}} \underline{\tilde{A}}) \underline{\tilde{x}}$$ (where $\underline{\tilde{A}} = \underline{\tilde{A}}_o + \delta\underline{\tilde{A}}$ as defined in equation (A4))

$$= -\underline{\tilde{x}}^T \delta\underline{\tilde{Q}} \underline{\tilde{x}} \quad \text{(from (A12))} \tag{A65}$$

$< 0 \qquad \underline{x} \neq 0$ since $\delta\underline{\tilde{Q}}$ is guaranteed to be $> \underline{0}$ in each of the propositions 1 to 5, according to Lemma 4.

Hence, η satisfies the stability theorem. ■

Proposition 5 is proved.

APPENDIX C

SOME INVESTIGATIONS OF HYBRID SYSTEMS

by

Alex Kostovetsky

Submitted to the Department of Mechanical Engineering on May 11, 1979 in partial fulfillment of the requirements for the Degree of Master of Science.

ABSTRACT

The purpose of this thesis was to investigate some inherent properties of hybrid systems. These systems include both continuous in time and discrete parts and have a particular importance in design and implementation of various digital control algorithms. In particular, problems of hybrid approximation for continuous "nominal" system and robustness of hybrid systems are studied.

The robustness problem for general control systems has been studied by M. Safonov but his results cannot be applied directly to hybrid systems in order to determine a critical value of the sampling interval which assures the system robustness. The problem is investigated in the thesis as well.

The practical three-dimensional control system is shown in order to illustrate the general relationships obtained for hybrid systems.

Thesis Supervisor: Professor Gunter Stein,
Department of Electrical Engineering
and Computer Science

ACKNOWLEDGMENTS

I am very grateful to Professor Gunter Stein for the constant encouragement, moral support and guidance he offered me during various stages of this thesis.

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INTRODUCTION

The subject of this thesis is investigation of some properties of hybrid systems. Such systems include both continuous (plant) and discrete (digital computer) parts. Inherent properties of hybrid structures are of particular interest when designing digital and analog devices to be used in closed-loop systems (control systems, for instance).

The first section provides a general description of hybrid systems, their components and introduces some notations.

In the second section a continuous time representation is shown for the hybrid system. Examples given in the section explain general features of a hybrid structure.

The third section of the thesis deals with properties of the induced norm of the hybrid operator. Both lower and upper bounds for that norm are derived. Their dependence on the sampling averaging interval is clarified as well as their impulse-like behavior.

A hybrid approximation of continuous operators is considered in the fourth section. An optimal approximation criterion is discussed and interpreted. The optimal coefficients of the hybrid approximation are derived for a sufficiently large class of linear continuous operators. Possible structural simplifications both in sampler and hold circuits are discussed and the optimal approximation for these situations are derived. Some examples are shown in the end of the section.

The fifth section of the thesis deals with the robustness problem for hybrid systems. This problem has been solved in general for various control systems but either for continuous or for discrete case. Those results are used to develop an appropriate sufficient stability conditions for hybrid

systems. The approach suggested for this purpose provides the value of sampling rate which preserves stability of a nominal continuous time system.

The last section describes an example of the concepts and methods developed in previous sections. A three-dimensional single input-single output closed loop control system is considered. A sampling rate which assures stability of the corresponded hybrid system is found by the suggested method. The optimal approximation hybrid system is constructed and compared with alternate hybrid system. To compare their performance when subject to noise, both systems have been simulated on a computer. Results of the simulation are discussed.

SECTION 1

HYBRID SYSTEMS

Consider two general linear operators

$$v(t) = \int_0^t G(t,\theta)u(\theta)d\theta \tag{1.1}$$

$$v_k(t) = \sum_{\ell=1}^{k} G_{k\ell}(t)\xi_\ell \tag{1.2}$$

where $u(\theta)$, $v(t)$, ξ_ℓ, $v_k(t)$ are vectors of dimensions M, N, L, N, respectively, $G(t,\theta)$ is an NxM matrix and $G_{k\ell}(t)$ is an LxM matrix.

The operator (1.1) represents a physical continuous time system, while (1.2) is a system which includes both digital and analog components. Systems of this type will be called "hybrid systems". Usually, they have the following structure.

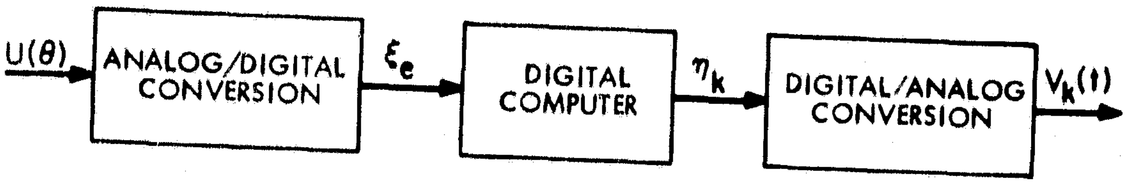

Fig. 1 Hybrid System

As seen in the Fig. 1, both input and output of a hybrid system are continous time signals. Such systems may be connected directly with various continuous time plants for different purposes.

A hybrid system consists of three parts: sampler, computer and hold circuit.

a) Sampler is a device which converts analog signals to a sequence of numbers (vector-valued) getting to a computer.

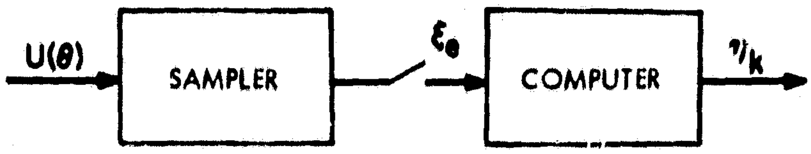

Fig. 2. Sampling Operation

The input signal $u(\theta)$ is averaged in some way over time interval τ so that each τ seconds a new value of ξ_ρ gets to the computer. This operation over a single time interval from $(\ell-1)\tau$ to $\ell\tau$ may be represented as

$$\xi_\ell = \int_{(\ell-1)\tau}^{\ell\tau} f_\ell(\theta)u(\theta)d\theta \tag{1.3}$$

where $f_\ell(\theta)$ is a certain matrix valued function, $u(\theta)$ and ξ_ℓ are vectors.

We define $f_\ell(\theta)$ so that

$$f_\ell(\theta) = 0 \text{ if } \theta \notin [\tau(\ell-1), \tau\ell).$$

b) Digital Computer. This block performs transformation of input sequence $\{\xi_\ell\}$, $\ell = 1,2,\ldots$ into output sequence $\{\eta_\kappa\}$, $\kappa = 0,1\ldots$. In case of linear realizable system it may be written as

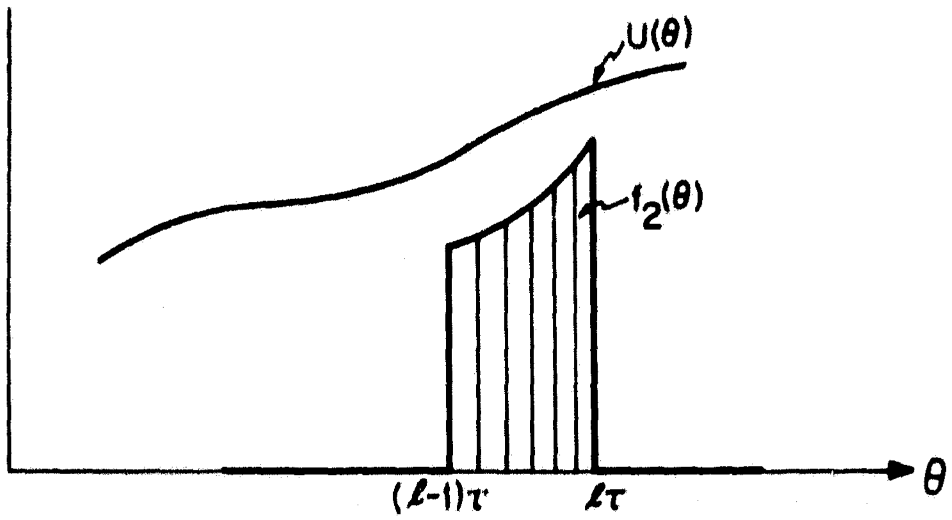

Fig. 3. Sampling

$$\eta_k = \sum_{\ell=1}^{k} D_{k\ell}\xi_\ell \tag{1.4}$$

where ξ_ℓ is an L-vector, η_k is an L_1-vector, and $D_{k\ell}$ is an L_1xL matrix sequence.

c) Hold Circuit. The input of this device is the sequence $\{\eta_k\}$ which can be multiplied by some continuous function $g_k(t)$ over each interval $[k\tau, (k+1)\tau)$ to produce the ultimate output of the hybrid system $v_k(t)$.

$$v_k(t) = g_k(t) \cdot \eta_k$$

where $g_k(t)$ is a NxL_1 matrix and is also defined to be zero outside the interval $[k\tau, (k+1)\tau)$.

Finally, the overall hybrid system which transforms $u(\theta)$ to $v_k(t)$ may be presented in the form

$$v_k(t) = g_k(t) \sum_{\ell=1}^{k} D_{k\ell}\xi_\ell, \quad k = 0,1\ldots \tag{1.5}$$

where

$$\xi_\ell = \int_{(\ell-1)\tau}^{\ell\tau} f_\ell(\theta)u(\theta)d\theta$$

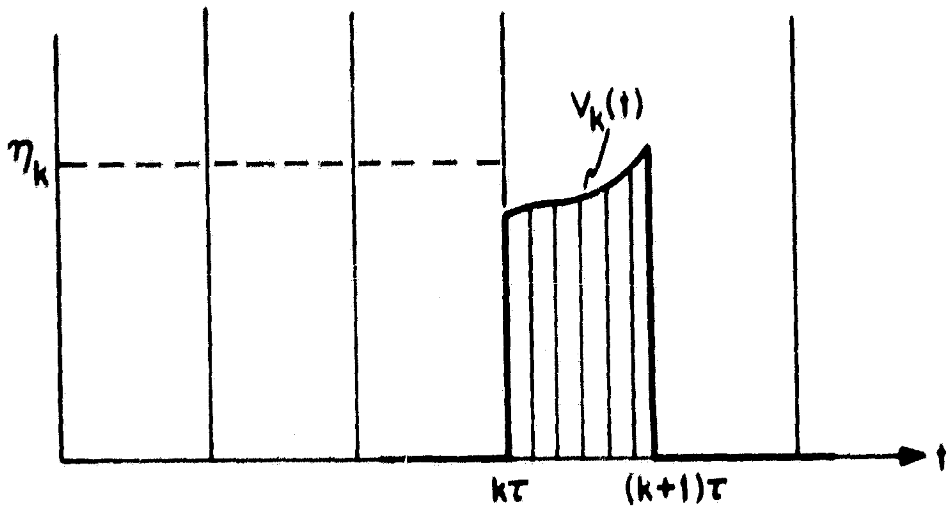

Fig. 4. Hold Operation.

SECTION 2

Hybrid Operator Representation

In this section we show how the hybrid system may be presented in a continuous time operator form. This representation allows us to investigate some specific norm properties of general hybrid operators and also is helpful in applying methods of continuous time systems to hybrid controllers.

The transformation (1.5) may be written in a continuous time form

$$v(t) = \int_0^t G(t,\theta)u(\theta)d\theta \tag{2.1}$$

if we introduce the function

$$G(t,\theta) = g_k(t) \sum_{\ell=1}^{k} D_{k\ell} f_\ell(\theta) \tag{2.2}$$

and let $k = \left[\frac{t}{\tau}\right]$ be in the integer part of $\frac{t}{\tau}$. This is verified by using expression (2.2) in the formula (2.1). We have

$$\begin{aligned} v(t) &= \int_0^t g_k(t) \sum_{\ell=1}^{k} D_{k\ell} f_\ell(\theta)u(\theta)d\theta = \\ &= g_k(t) \sum_{\ell=1}^{k} D_{k\ell} \int_0^{k\tau} f_\ell(\theta)u(\theta)d\theta = \\ &= g_k(t) \sum_{\ell=1}^{k} D_{k\ell} \int_{(\ell-1)\tau}^{\ell\tau} f_\ell(\theta)\, u(\theta)d\theta = \\ &= g_k(t) \sum_{\ell=1}^{k} D_{k\ell}\xi_\ell = v_k(t),\ k = 0,1\ldots \end{aligned} \tag{2.3}$$

The result shows that any hybrid operator (1.5) may be presented in a form (2.1) with a weighting function (2.2). Also, one can see from the expression (2.2) that the hybrid operator is inherently factorized into two factors: one depends only on t, the other only on θ.

Usually, shifted versions of the same function are used both in sampling and hold devices, i.e.

$$
\begin{aligned}
g_m(t) &= g_0(t-m\tau) \\
f_\ell(\theta) &= f_0(\theta-\ell\tau)
\end{aligned}
\tag{2.4}
$$

where m and ℓ are arbitrary integers, and $g_0(\lambda)$ and $f_0(\lambda)$ are given sample and hold functions.

Example.

Consider the following example of a scalar hybrid system

$$
\begin{aligned}
&\text{Sampler:} \quad f_0(\lambda) = \begin{cases} \frac{1}{\varepsilon} & -\varepsilon < \lambda \leq 0 \\ 0 & \text{otherwise} \end{cases} \\
&\text{Computer:} \quad D_{k\ell} = \mu^{k-\ell} \\
&\text{Hold:} \quad g_0(\lambda) = 1, \quad 0 \leq \lambda < \tau
\end{aligned}
\tag{2.5}
$$

Formula (1.5) yields for $G(t,\theta)$:

$$
G(t,\theta) = \sum_{\ell=1}^{k} \mu^{k-\ell} \xi_\ell
\tag{2.6}
$$

where

$$\zeta_0 = \frac{1}{\varepsilon} \int_{-\varepsilon}^{0} u(\theta)\,d\theta$$

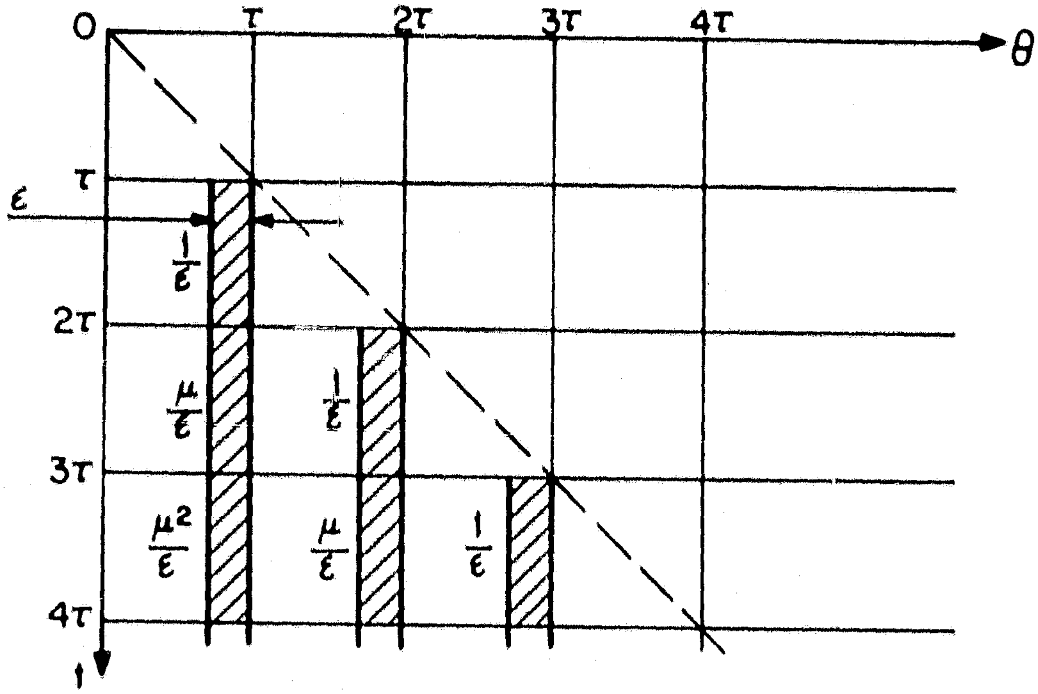

Fig. 5. Operator $G(t,\theta)$

Fig. 5 shows a structure of the function $G(t,\theta)$. In this particular case functions $g_0(\lambda)$ and $f_0(\lambda)$ are shown in Fig. 6. They represent so called "zero order hold" and approximate "impulsive sampling", respectively [1].

Another possible example of a sampling function is

$$f_0(\lambda) = \frac{1}{c}\, e^{-\alpha\lambda} \tag{2.7}$$

where

$$c = \frac{1}{\alpha}\,(e^{\alpha\tau} - 1)$$

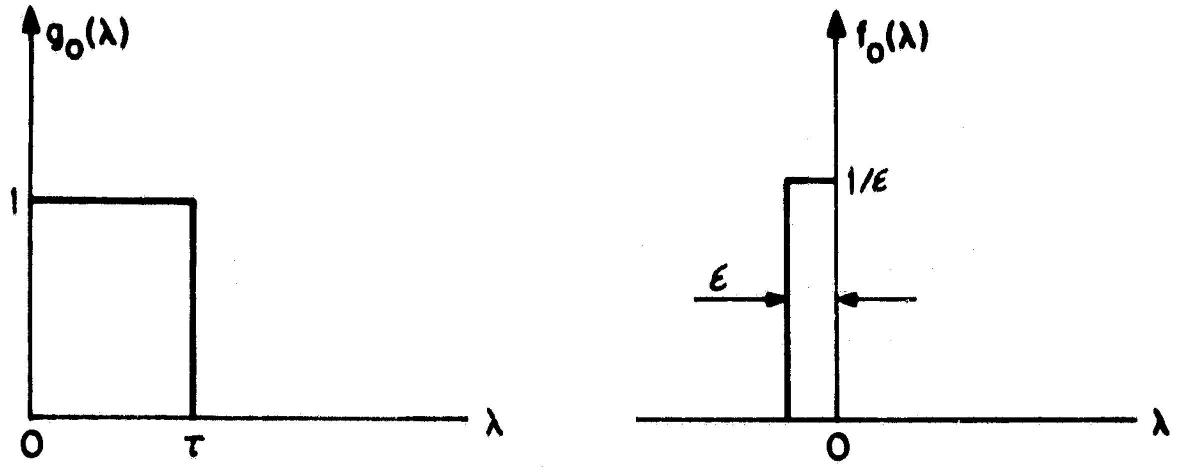

Fig. 6. Example of Sampling and Hold Functions

This is called exponentially weighted sampling and also approximates impulsive sampling for α sufficiently large.

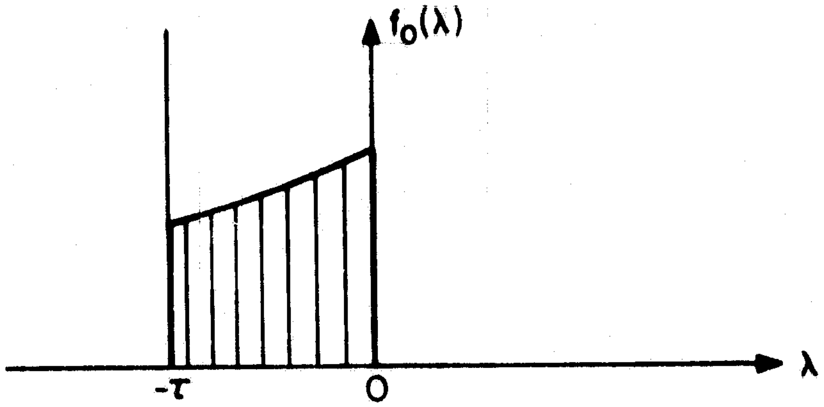

Fig. 7. Exponentially Weighted Sampling

In more general situation functions $f_0(\lambda)$ and $g_0(\lambda)$ are not necessarily scalars. For example, the samples might be

$$f_0(\lambda) = \begin{pmatrix} f_{10}(\lambda) \\ f_{20}(\lambda) \end{pmatrix}$$

where $f_{10}(\lambda)$ and $f_{20}(\lambda)$ are scalar functions shown on Fig. 8.

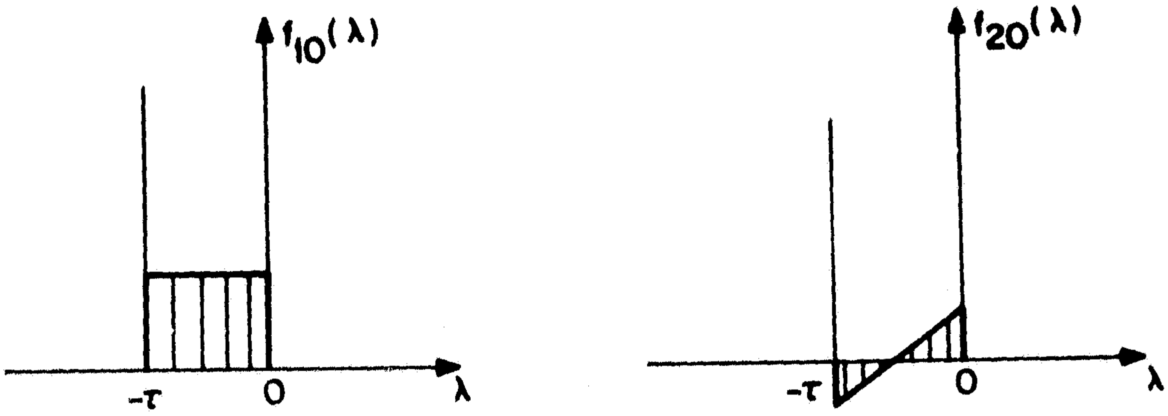

Fig. 8. Example of Multivariable Sampling

$$f_{10}(\lambda) = \begin{cases} 1 & -\tau < \lambda \leq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$f_{20}(\lambda) = \begin{cases} \beta\left(0 + \frac{\tau}{2}\right) & -\tau < \lambda \leq 0 \\ 0 & \text{otherwise} \end{cases} \tag{2.8}$$

In this case f_{10} samples the average value and samples the average slope of the input function. Similarly, the hold circuit may have the form

$$g_0(\lambda) = (g_{10}(\lambda),\ g_{20}(\lambda))$$

where $g_{10}(\lambda)$ and $g_{20}(\lambda)$ are, for example

$$
g_{10}(\lambda) = \begin{cases} 1, & 0 \leq \lambda < \tau \\ 0, & \text{otherwise} \end{cases}
$$

$$
g_{20}(\lambda) = \begin{cases} \gamma\lambda, & 0 \leq \lambda < \tau \\ 0, & \text{otherwise} \end{cases} \tag{2.9}
$$

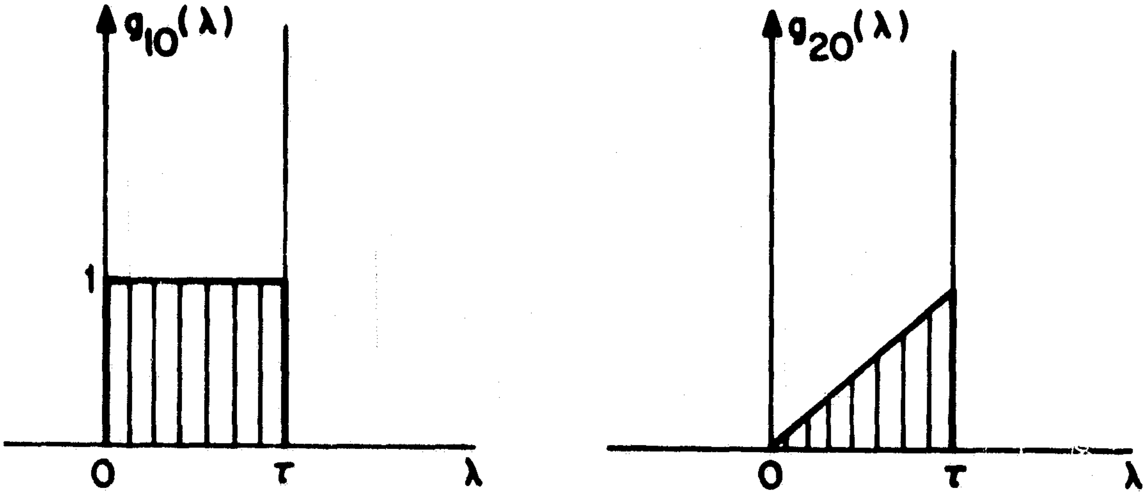

Fig. 9. Example of Multivariable Hold

Here, g_{10} is a zero order hold and g_{20} is an ideal first order hold. The computer can then command both the level and the slope of the output function.

SECTION 3

Norm Inequalities for Hybrid Operators

Now we establish upper and lower bounds for induced norm of the hybrid operator.

A norm of operator G is defined as [2]

$$||G|| = \sup_{u} \frac{||Gu||}{||u||} \tag{3.1}$$

where $||u||$ is a Euclidean norm of a vector function

$$||u||^2 = \int_0^\infty u^T(\theta)u(\theta)d\theta \tag{3.2}$$

In the case of our hybrid operator we have

$$(Gu)(t) = g_k(t) \sum_{\ell=1}^{k} D_{k\ell}\xi_\ell \tag{3.3}$$

where

$$\xi_\ell = \int_{(\ell-1)\tau}^{\ell\tau} f_\ell(\theta)u(\theta)d\theta$$

with $g_k(t)$, $D_{k\ell}$, $f_\ell(\theta)$ are matrices and ξ_θ is L, $u(\theta)$ is M dimensional vectors.

For such operators we can write

$$||G||^2 = \sup_{u} \frac{||Gu||^2}{||u||^2} = \tag{3.4}$$

$$= \sup_{u} \frac{1}{||u||^2} \int_0^\infty (Gu)^T(Gu)dt$$

$$\int_0^{\infty}(Gu)^T(Gu)dt = \sum_{k=0}^{\infty}\int_{k\tau}^{(k+1)\tau}(Gu)^T(Gu)dt =$$

$$= \sum_{k=0}^{\infty}\int_{k\tau}^{(k+1)\tau} tr[g_k(t)\sum_{\ell=1}^{k} D_{k\ell}\xi_\ell \cdot \sum_{m=1}^{k}\xi_m^T D_{km}^T g_k^T(t)]dt \tag{3.5}$$

Define

$$\Delta_k \triangleq \int_{k\tau}^{(k+1)\tau} g_k^T(t)g_k(t)dt \tag{3.6}$$

Then,

$$||Gu||^2 = \sum_{k=0}^{\infty} tr[\Delta_k \sum_{\ell=1}^{k}\sum_{m=1}^{k} D_{km}\xi_m\xi_\ell^T D_{k\ell}^T \le$$

$$\le \sum_{k=0}^{\infty} tr[\Delta_k] \cdot \sum_{\ell=1}^{k}\sum_{m=1}^{k} tr[D_{km}\xi_m\xi_\ell^T D_{k\ell}^T] \le$$

$$\le \sum_{k=0}^{\infty} tr[\Delta_k] \cdot \sum_{\ell=1}^{k}\sum_{m=1}^{k} tr[D_{k\ell}^T D_{km}] \cdot tr[\xi_m\xi_\ell^T] \tag{3.7}$$

Due to properties of a trace, we have

$$(tr[\xi_m\xi_\ell^T])^2 \le ||\xi_m||^2 \cdot ||\xi_\ell||^2 \tag{3.8}$$

and

$$\mathrm{tr}[\xi_m \xi_\ell^T] \leq ||\xi_m|| \cdot ||\xi_\ell|| \tag{3.9}$$

So,

$$||Gu||^2 \leq \sum_{k=0}^{\infty} \mathrm{tr}[\Delta_k] \cdot \sum_{\ell=1}^{k} \sum_{m=1}^{k} \mathrm{tr}[D_{k\ell}^T D_{km}] \cdot ||\xi_m|| \cdot ||\xi_\ell|| \tag{3.10}$$

Properties of matrix traces used in (3.7) - (3.10) are given in Appendix II.

Quantities $\mathrm{tr}[\Delta_k]$ and $\mathrm{tr}[D_{k\ell}^T D_{km}]$ are independent of $u(\theta)$. Further in this section we will restrict ourselves to the sampling function

$$f_0(\lambda) = \begin{cases} \frac{1}{\varepsilon}\rho & -\varepsilon \leq \lambda < 0 \\ 0 & \text{otherwise} \end{cases} \tag{3.11}$$

where ρ is a constant LxM matrix in order to obtain a simple result. However, as $\varepsilon \to 0$ the result is still true no matter what the shape of the function $f_0(\lambda)$ is inside the interval $[-\varepsilon, 0)$.

Under these assumptions, we get

$$\sup_u \frac{||\xi_\ell||^2}{||u||^2} = \sup_u \frac{1}{||u||^2} \left|\left| \int_{\tau(\ell-1)}^{\tau\ell} f_0(\theta - \tau\ell) u(\theta) d \right|\right|^2 =$$

$$= \sup_u \frac{1}{||u||^2} \sum_{i=1}^{L} \left(\int_{\tau\ell-\varepsilon}^{\tau\ell} \sum_{j=1}^{M} \frac{1}{\varepsilon} \rho_{ij} u_j(\theta) d\theta \right)^2 =$$

$$= \sup_u \frac{1}{||u||} \sum_{i=1}^{L} \left(\frac{1}{\varepsilon} \sum_{j=1}^{M} \rho_{ij} \int_{\tau\ell-\varepsilon}^{\tau\ell} u_j(\theta) d\theta \right)^2 \leq$$

$$\leq \sup_u \frac{1}{||u||^2} \cdot \frac{M}{\varepsilon} \sum_{i=1}^{L} \sum_{j=1}^{M} \rho_{ij}^2 \left(\int_{\tau\ell-\varepsilon}^{\tau\ell} u_j(\theta) d\theta \right)^2 \leq$$

$$\leq \sup_u \frac{1}{||u||^2} \cdot \frac{M}{\varepsilon^2} \cdot \sum_{i=1}^{L}\sum_{j=1}^{M} \varepsilon \, \rho_{ij}^2 \int_{\tau\ell-\varepsilon}^{\tau\ell} u_j^2(\theta)\,d\theta =$$

$$= \frac{r^2 LM}{\varepsilon} \sup_u \frac{1}{||u||} \sum_{j=1}^{M} \int_{\tau\ell-\varepsilon}^{\tau\ell} u_j^2(\theta)\,d\theta \qquad (3.12)$$

where $r \triangleq \max_i \max_j \rho_{ij}$.

The expression (3.12) may be simplified because

$$\sum_{j=1}^{M} \int_{\tau\ell-\varepsilon}^{\tau\ell} u_j^2(\theta)\,d\theta = \sum_{j=1}^{M} \int_{-\varepsilon}^{0} u_j^2(\theta-\tau\ell)\,d\theta =$$

$$= \int_{-\varepsilon}^{0} u^T(\theta-\tau\ell)\,u(\theta-\tau\ell)\,d\theta \qquad (3.13)$$

Then,

$$\sup_u \frac{\int_{-\varepsilon}^{0} u^T(\theta-\tau\ell)\,u(\theta-\tau\ell)\,d\theta}{\int_{0}^{\infty} u^T(\theta)\,u(\theta)\,d(\theta)} = 1 \qquad (3.14)$$

for $\ell = 1,2,\ldots$. Therefore, for the hybrid operator norm we have from (3.10)

$$||G||^2 \leq \sum_{k=0}^{\infty} tr[\Delta_k] \cdot \sum_{\ell=1}^{k}\sum_{m=1}^{k} tr[D_{k\ell}^T D_{km}] \cdot \frac{r^2 LM}{\varepsilon} \leq \frac{A_1^2}{\varepsilon} \qquad (3.15)$$

(3.15)

and

$$||G|| \leq \frac{A_1}{\sqrt{\varepsilon}} \tag{3.16}$$

We note that

$$A_1 = \sum_{k=0}^{\infty} tr[\Delta_k] \sum_{\ell=1}^{k} \sum_{m=1}^{k} tr[D_{k\ell}^T D_{km}]$$

is not necessarily finite. One could conceive sequences Δ_k and $D_{k\ell}$ so that the upper bound of the hybrid operator is infinite

In order to obtain a lower bound for the induced norm of the hybrid operator we may select any particular input $u(\theta)$ and consider the corresponding value of

$$\frac{||Gu||}{||u||}$$

as a lower bound of the operator norm.

We select

$$u(\theta) = \begin{cases} \beta & \tau-\varepsilon \leq \theta < \tau \\ 0, & \text{otherwise} \end{cases}$$

where β is a constant M-vector.

Then, for $(Gu)(t)$ we obtain

$$(Gu)(t) = g_k(t) \sum_{\ell=1}^{k} D_{k\ell}\, \xi_\ell \tag{3.17}$$

where

$$\xi_\ell = \int_{\tau\ell-\varepsilon}^{\tau\ell} f_\ell(\theta) u(\theta) d\theta = \begin{cases} \rho\,\beta, & \ell = 1 \\ 0, & \text{otherwise} \end{cases}$$

and therefore,

$$(Gu)(t) = [g_k(t) \sum_{\ell=1}^{k} D_{k\ell}] \rho\beta = g_k(t) D_{k0} \rho\beta \tag{3.18}$$

$$||Gu||^2 = \sum_{k=0}^{\infty} tr[\Delta_k] \cdot tr[D_{k0}^T D_{k0}] \cdot ||\rho\beta||^2 \tag{3.19}$$

$$||u||^2 = \int_0^{\infty} u^T(\theta) u(\theta) d\theta = \varepsilon \, ||\beta||^2 \tag{3.20}$$

Then,

$$\frac{||Gu||^2}{||u||^2} = \frac{A_2^2}{\varepsilon}\,, \qquad ||G|| \geq \frac{A_2}{\sqrt{\varepsilon}} \tag{3.21}$$

The conclusion of this analysis is that the norm of a hybrid operator with interval sampling becomes unbounded as the interval sampling becomes unbounded as the interval vanishes (tends toward impulsive sampling). This means that the impulsive sampling has this specific norm property in spite of the fact it is widely used as a simplest model of the sampling operation [1].

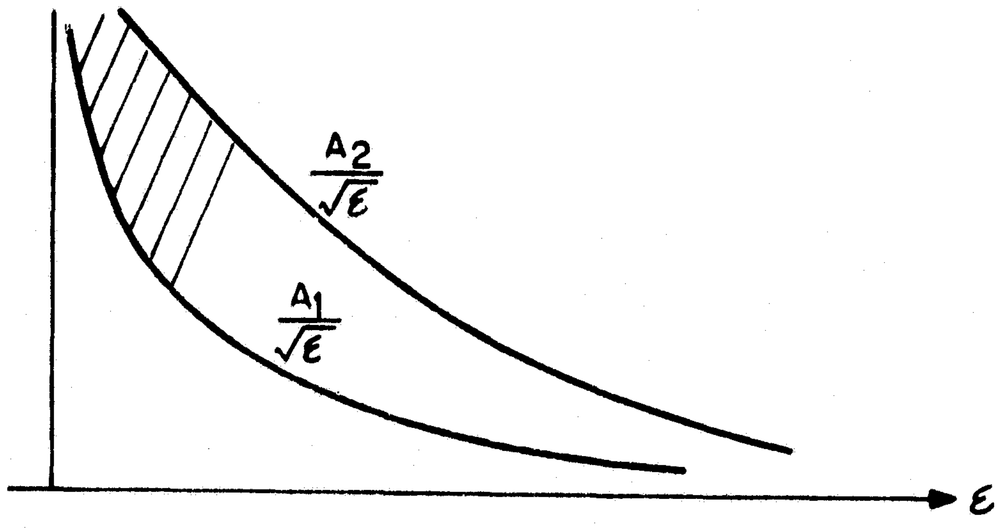

Fig. 10. Norm Bounds for the Hybrid Operator

SECTION 4

Hybrid Approximation of Continuous Operators

As we have shown in Section 2 the hybrid operator in general, may be represented in a form

$$G(t,\theta) = g_0(t-k\tau)\sum_{\ell=1}^{k} D_{k\ell} f_0(\theta-\tau\ell) \tag{4.1}$$

where k is the integer part of $\frac{t}{\tau}$, and

$$(Gu)(t) = \int_0^t G(t,\theta)u(\theta)d\theta =$$

$$= \int_0^t g_0(t-k\tau)\sum_{\ell=1}^{k} D_{k\ell} f_0(\theta-\tau\ell)u(\theta)d\theta =$$

$$= g_0(t-k\tau)\sum_{\ell=1}^{k} D_{k\ell}\int_{\tau(\ell-1)}^{\tau\ell} f_0(\theta-\tau\ell)u(\theta)d\theta =$$

$$= g_0(t-k\tau)\sum_{\ell=1}^{k} D_{k\ell}\xi_\ell \tag{4.2}$$

The problem of optimal approximation of a continuous time linear operator $\overline{G}(t,\theta)$ by the hybrid operator $G(t,\theta)$ may be formulated as follows:

Find the structure of a hybrid system, i.e. matrices $g_0(\lambda)$, $D_{k\ell}$, $f_0(\lambda)$ such that

$$J = \lim_{T\to\infty}\frac{1}{T}\int_0^T dt \int_0^t \mathrm{tr}[(G(t,\theta) - \overline{G}(t,\theta))^T(G(t,\theta) - \overline{G}(t,\theta))]d\theta \tag{4.3}$$

is minimized.

Physically we can interpret the criterion as a mean square deviation of signals $v(t)$ and $v_k(t)$ produced by the nominal and hybrid systems, provided that both are driven by the same white noise process.

To prove this we note that the two system responses are:

$$v(t) = \int_0^t \overline{G}(t,\theta)u(\theta)d\theta$$

$$v_k(t) = \int_0^t G(t,\theta)u(\theta)d\theta$$

Then,

$$v_k(t) - v(t) = \int_0^t [G(t,\theta) - \overline{G}(t,\theta)]u(\theta)d\theta \tag{4.4}$$

$$E[(v_k(t) - v(t))(v_k(t) - v(t))^T] =$$

$$= E\left[\int_0^t (G(t,\theta) - \overline{G}(t,\theta))u(\theta)d\theta \cdot \int_0^t u^T(\omega)(G(t,\omega) - G(t,\omega))^T d\omega\right] =$$

$$= E\left\{\int_0^t \int_0^t d\theta d\omega[(G(t,\theta) - \overline{G}(t,\theta))(G(t,\omega) - \overline{G}(t,\omega))^T]u(\theta)u^T(\omega) =\right.$$

$$= \int_0^t \int_0^t d\theta d\omega[(G(t,\theta) - \overline{G}(t,\theta))(G(t,\omega) - \overline{G}(t,\omega))E[U(\theta)u^T(\omega)] =$$

$$= \int_0^t (G(t,\theta) - \overline{G}(t,\theta))(G(t,\theta) - \overline{G}(t,\theta))^T d\theta \tag{4.5}$$

A natural measure of total deviation between two time variant random vectors is

$$J = E[\lim_{T\to\infty} \frac{1}{T} \int_0^T tr[(v_k(t) - v(t))(v_k(t) - v(t))^T]dt \tag{4.6}$$

Subsituting (4.5) into (4.6) we obtain the criterion (4.3)

Perfectly matched hold

Suppose now that the nominal continuous operator is such that

$$\overline{G}(t,\theta) = \overline{H}(t)\overline{S}(\theta) \tag{4.7}$$

where $\overline{H}(t)$ satisfies a functional equation

$$\overline{H}(t_1 + t_2) = \overline{H}(t_1) \cdot \overline{M}(t_2) \tag{4.8}$$

for some matrix $\overline{M}(t)$ of dimension KxK. We can then look for optimal approximation matrices in the form

$$D_{k\ell} = M_k \cdot d_\ell$$

where the M_k's are KxK and d_ℓ's are KxL matrices.

The hybrid operator (4.1) then takes the form

$$G(t,\theta) = g_0(t - k\tau)M_k \sum_{\ell=1}^{k} d_\ell f(\theta - \tau\ell)$$

and we can identify $g_0(t-k\tau)$ and M_k as

$$\begin{aligned} g_0(t-k\tau) &= \overline{H}(t - k\tau) \\ M_k &= \overline{M}(k\tau) \end{aligned} \tag{4.9}$$

This identification means that the t-dependent factor of the nominal operator may be duplicated exactly by the hybrid system for all t. Now call

Now call,
$$s(\theta) = \begin{cases} 0, & \theta > k\tau \\ \sum_{\ell=1}^{k} d_\ell f_0(\theta-\tau\ell), & \theta \leq k\tau \end{cases}$$

and substitute into the criterion (4.3).

$$J = \lim_{T\to\infty} \frac{1}{T} \int_0^T dt \int_0^t \mathrm{tr}[(s(\theta) - \bar{s}(\theta))^T H^T(t) H(t)(s(\theta) - \bar{s}(\theta))]d\theta =$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_0^T dt \int_0^t \mathrm{tr}[\bar{H}^T(t)H(t)(s(\theta) - \bar{s}(\theta))(s(\theta) - \bar{s}(\theta))^T]d\theta =$$

$$= \mathrm{tr}[\lim_{T\to\infty} \frac{1}{T} \int_0^T H^T(t)H(t)dt \int_0^{k\tau} (s\theta - \bar{s}(\theta))(s(\theta) - \bar{s}(\theta))^T d\theta +$$

$$+ \int_{k\tau}^{t} \bar{s}(\theta)\bar{s}^T(\theta)d\theta\}] =$$

$$= \mathrm{tr}[\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N} \int_{k\tau}^{(k+1)\tau} \bar{H}^T(t)H(t)dt \cdot \sum_{\ell=1}^{k} \int_{(\ell-1)\tau}^{\ell\tau} (s(\theta) - \bar{s}(\theta)(s(\theta) - \bar{s}(\theta))^T d\theta] +$$

$$= \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N} \mathrm{tr}[\int_{k\tau}^{(k+1)\tau} H^T(t)H(t)dt \cdot$$

$$\cdot \sum_{\ell=1}^{k} \int_{(\ell-1)\tau}^{\ell\tau} (s(\theta) - \bar{s}(\theta))(\bar{s}(\theta) - \bar{s}(\theta))^T d\theta] + \tilde{J} \qquad (4.11)$$

where

$$\tilde{J} = \mathrm{tr}(\lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N} \int_{k\tau}^{(k+1)\tau} \bar{H}^T(t)\bar{H}(t)dt \cdot \int_{k\tau}^{t} \bar{s}(\theta)\bar{s}(\theta)^T d\theta]$$

$$J - \tilde{J} = \lim_{N\to\infty} \frac{1}{N}\sum_{k=0}^{N} \mathrm{tr}[B_k \sum_{\ell=1}^{k} \int_{(\ell-1)\tau}^{\ell\tau} (s(\theta) - \overline{s}(\theta))(s(\theta) - \overline{s}(\theta))^T d\theta]$$

where

$$B_k \triangleq \int_{k\tau}^{(k+1)\tau} \overline{H}^T(t)\overline{H}(t)dt \tag{4.12}$$

Since each term of this double sum

$$J_{k\ell} = \mathrm{tr}[B_k \int_{(\ell-1)\tau}^{\ell\tau} (s(\theta) - s(\theta))(\overline{s}(\theta) - \overline{s}(\theta))^T d\theta] \tag{4.13}$$

depends only on its own matrix of coefficients d_ℓ and not on other $d_m (m \neq \ell)$, it can be minimized independently over d_ℓ. It corresponds to a single square optimization shown on Fig. 11.

$$J_{k\ell} = \mathrm{tr}[B_k \int_{(\ell-1)\tau}^{\ell\tau} (s(\theta)s^T(\theta) - 2s(\theta)\overline{s}^T(\theta) + \overline{s}(\theta)\overline{s}^T(\theta))d\theta$$

$$s(\theta) = d_\ell f_0(\theta - \tau\ell) \qquad \text{for } \tau(\ell-1) \leq \theta \leq \tau\ell$$

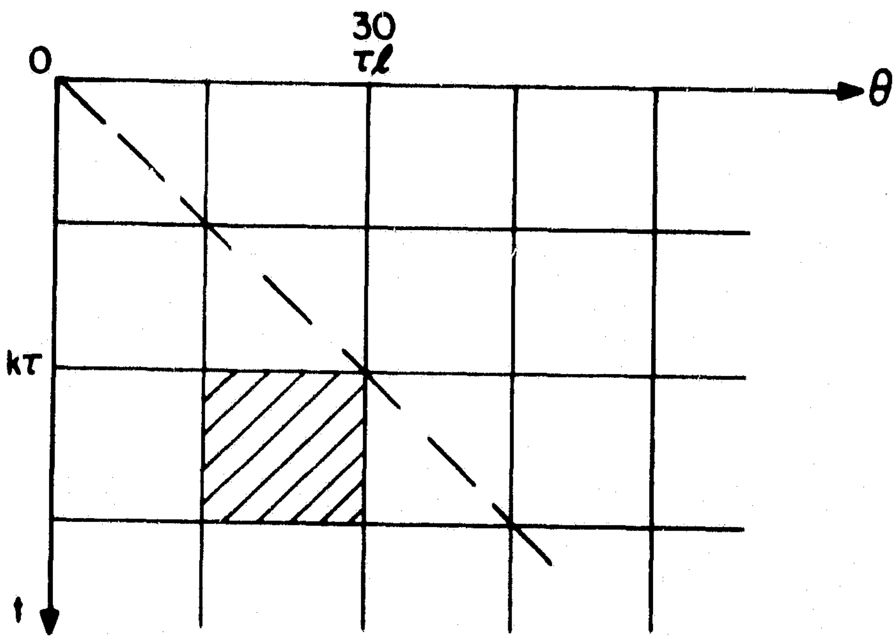

Fig. 11 Optimization Square

So,

$$J_{k\ell} = \mathrm{tr}[B_k \int_{(\ell-1)\tau}^{\ell\tau} (d_\ell f_0(\theta-\tau\ell) f_0^T(\theta-\tau\ell) d_\ell^T -$$

$$- 2d_\ell f_0(\theta-\tau\ell)\bar{S}^T(\theta) + \bar{S}(\theta)\bar{S}^T(\theta))d\theta] \tag{4.14}$$

Introduce following notations

$$F_\ell \triangleq \int_{(\ell-1)\tau}^{\ell\tau} f_0(\theta-\tau\ell) f_0^T(\theta-\tau\ell)d\theta \equiv F_0 \tag{4.15}$$

$$\phi_\ell \triangleq \int_{(\ell-1)\tau}^{\ell\tau} \bar{S}(\theta) f^T(\theta-\tau\ell)d\theta$$

Then,

$$J_{k\ell} = \mathrm{tr}[B_k(d_\ell F_\ell d_\ell^T - 2d_\ell \phi_\ell^T + \int_{\tau(\ell-1)}^{\tau\ell} \bar{S}(\theta)\bar{S}^T(\theta)d\theta]$$

Using formulas IA and IB of Appendix, we come to the equation

$$\frac{\partial J_{k\ell}}{\partial d_\ell} = B_k(d_\ell F_0 - \psi_\ell) = 0 \tag{4.16}$$

and

$$d_\ell = \phi_\ell F_0^{-1} \tag{4.17}$$

if F_0 and B_k are invertible. This implies that the optimal hybrid operator approximation is

$$G(t,\theta) = g_0(t - k\tau)M_k \int_{\ell=1}^{k} \phi_\ell F_0^{-1} f_0(\theta-\tau\ell) \tag{4.18}$$

where

$$g_0(t-k\tau) \cdot M_k = \overline{H}(t)$$

Consider now the special case

$$\overline{G}(t,\theta) = Ce^{A(t-\theta)}B \tag{4.19}$$

where A is NxN matrix. Then (4.7) - (4.8) imply that

$$\begin{aligned} \overline{H}(t) &= Ce^{At} \\ \overline{S}(\theta) &= e^{-A\theta}B \end{aligned} \tag{4.20}$$

and from (4.9):

$$M_k = e^{Ak\tau} \tag{4.21}$$

$$g_0(t-k\tau) = C\, e^{A(t-k\tau)} \tag{4.22}$$

Suppose, we also define

$$f_0(\theta) = e^{-A\theta}B \tag{4.23}$$

Then, for F_0 and ψ_ℓ we obtain by definition (4.15)

$$F_0 = \int_{-\tau}^{0} f_0(\theta) f_0^T(\theta)\, d\theta = \int_{-\tau}^{0} e^{-A\theta}\, BB^T e^{-A^T\theta} d\theta$$

$$\begin{aligned} \phi_\ell &= \int_{(\ell-1)\tau}^{\ell\tau} \overline{S}(\theta) f_0^T(\theta-\tau\ell)\, d\theta = \int_{-\tau}^{0} \overline{S}(\theta+\tau\ell) f_0^T(\theta)\, d\theta = \\ &= e^{-A\tau\ell} \int_{-\tau}^{0} e^{-A\theta}\, BB^T e^{-A^T\theta} d\theta = e^{-A\tau\ell} F_0 \end{aligned} \tag{4.24}$$

F_0 is full rank if (A,B) is controllable [1]. The formula (4.17) yields:

$$d_\ell = \phi_\ell F_0^{-1} = e^{-A\tau\ell} \tag{4.25}$$

and the corresponding value of $J_{k\ell}$ and, consequently $J-\tilde{J}$, is zero.

For the hybrid operator we then have

$$G(t,\theta) = g_0(t-k\ )M_k \sum_{\ell=1}^{k} d_\ell f_0(\theta-\tau\ell) =$$

$$= Ce^{At}e^{-A\tau k}e^{Ak\tau}\sum_{\ell=1}^{k} e^{-A\tau\ell}e^{-A\theta}e^{A\tau\ell}\ B = \tag{4.26}$$

$$= Ce^{At}\sum_{\ell=1}^{k} e^{-A\theta}B\ 1_\ell(\theta) = \begin{cases} C\ e^{A(t-\theta)}B, & \theta \le k\tau \\ 0 & \text{otherwise} \end{cases} \tag{4.27}$$

The function $1_\ell(\theta)$ is defined as

$$1_\ell(\theta) = \begin{cases} 1 & (\ell-1)\tau < \theta \le \tau\ell \\ 0, & \text{otherwise} \end{cases}$$

The output of the hybrid system is

$$(Gu)(t) = \int_0^t G(t,\theta)u(\theta)d\theta =$$

$$= \int_0^{k\tau} C\ e^{A(t-\theta)}Bu(\theta)d\theta = \tag{4.28}$$

$$= \int_0^{k\tau} \overline{G}(t,\theta)u(\theta)d\theta$$

This result means that a continuous time linear time invariant nominal operator may be exactly approximated by choosing appropriate sampling and hold structures, except for an inherent sampling error in the "triangle strip". This is illustrated in Fig. 12.

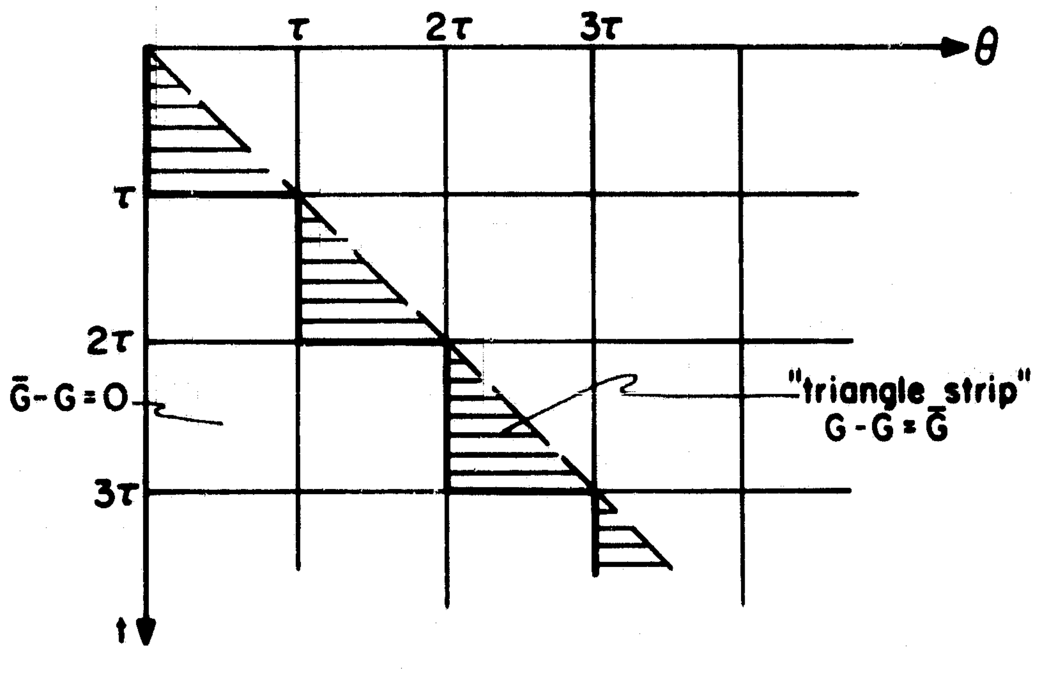

Fig. 12. The "triangle strip"

Note that the optimal sampling and hold circuits required by (4.22) and (4.23) are themselves multivariable linear systems of order N. The sampler takes the form

$$\xi_\ell = \int_{(\ell-1)\tau}^{\ell\tau} e^{A(\ell\tau-\theta)} Bu(\theta)d\theta$$

which has the block diagram in Fig. 13.

The hold circuit has the block diagram in the Fig. 14.

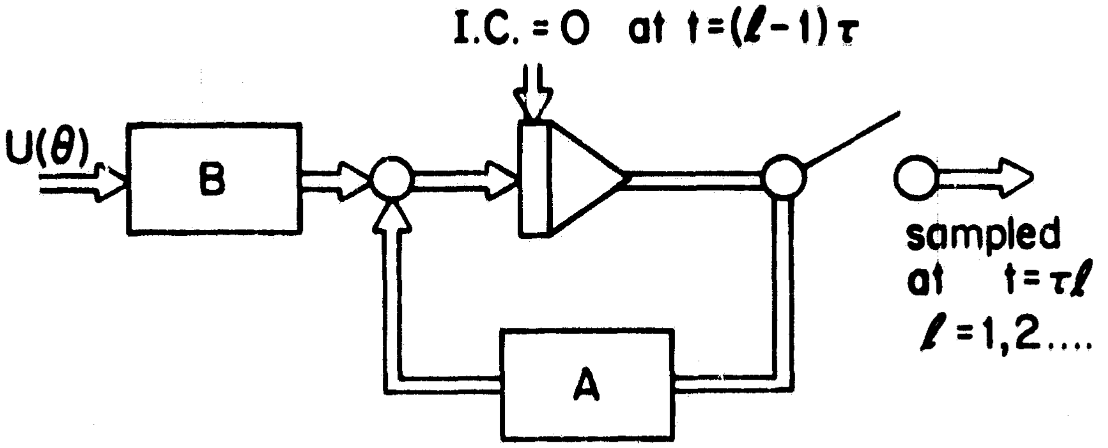

Fig. 13. Optimal Sampler

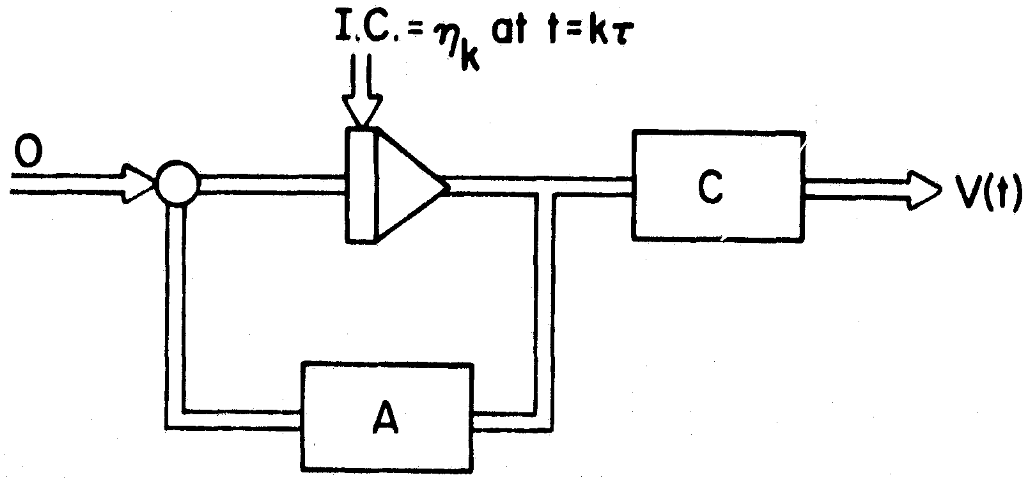

Fig. 14. Optimal Hold

Those circuits can be simplified by fixing the matrices $f_0(\lambda)$ and $g_0(\lambda)$ and using formula (4.15) to provide optimal value of matrices d_ℓ given the fixed analog structures.

Example

a) nominal system:

$$\overline{G}(t,\theta) = e^{A(t-\theta)}$$

b) exact match of hold, i.e.

$$g_0(t) = \overline{H}(t) = e^{At}$$
$$M_k = e^{Ak\tau} \tag{4.29}$$

c) $u(\theta) = f\gamma(\theta)$ (4.30)

where $\gamma(\theta)$ is a scalar function of time. f is a constant L-vector.

d) $$f_0(\theta) = \begin{cases} \frac{1}{\varepsilon} I & -\varepsilon \leq \theta < 0 \\ 0 & \text{otherwise} \end{cases}$$

where I is an LxL identity matrix. How we can apply formula (4.17) to obtain an optimal but not exact approximation of the nominal continuous operator.

$$F_0 = \int_{-\tau}^{0} f_0(\theta) f_0^T(\theta) d\theta = \frac{1}{\varepsilon} I \tag{4.31}$$

$$\phi_\ell = \int_{-\tau}^{0} \overline{S}(\theta+\tau\ell) f_0^T(\theta) d\theta =$$

$$= e^{-A\tau\ell} \int_{-\varepsilon}^{0} e^{-A\theta} \frac{1}{\varepsilon} d\theta \tag{4.32}$$

$$d_\ell = \phi_\ell F_0^{-1} = e^{-A\tau\ell} \int_{-\varepsilon}^{0} e^{-A\theta} d\theta \tag{4.33}$$

The simplified sampler structure for this case is shown in Fig. 15.

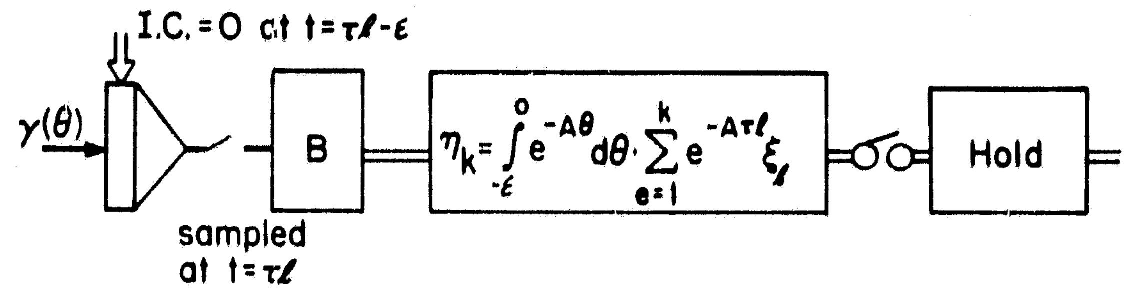

Fig. 15. Simplified Sampler

This is equivalent to the structure on Fig. 16.

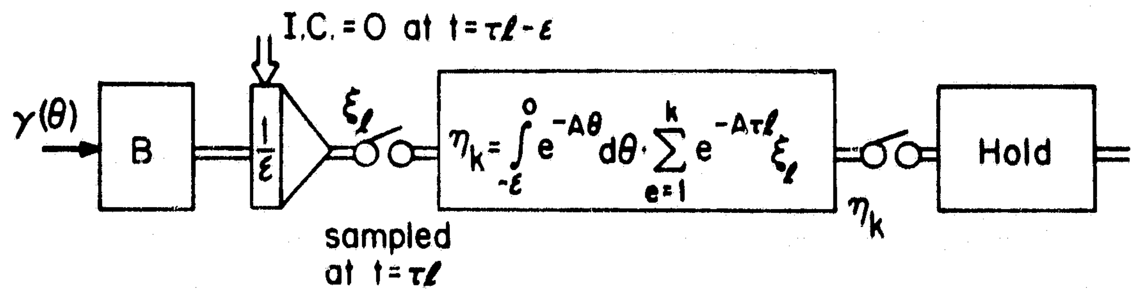

Fig. 16. Sampler of Reduced Dimension

Note that if ϵ goes to zero with finite τ $\eta_k \to 0$, which means that the impulsive sampling produces signals which should be neglected under the optimal hybrid approximation. This result is natural since the optimal approximation

approximation was designed to match outputs of two systems subject to the same white noise. But the impulsive sampling leads to an infinite mean square deviation between the two samples, so the output should be supporessed by choosing d_ℓ so that $\lim_{\varepsilon\to 0} d_\ell = 0$ in the optimal hybrid approximation.

Instead of the N-dimensional sampling circuit in Fig. 13 we now have a one-dimensional integrator in Fig. 16.

The formula (4.17) provides an "optimal" computational procedure (convolution) given an optimal hold circuit structure perfectly matched outside the "triangle strip".

<u>Perfectly matched sampler</u>

Consider now the dual case when the sampling operation is perfectly matched outside the "triangle strip" but the hold structure is unknown and to be determined.

Similar to the above case we suppose $\bar{S}(\theta)$ is such that

$$\bar{S}(\theta_1 + \theta_2) = \bar{d}(\theta_1) \cdot \bar{S}(\theta_2) \tag{4.36}$$

where $\bar{S}(\theta)$ is a KxM matrix.

Then,

$$\bar{S}(\theta) = \bar{S}(\theta-\tau\ell+\tau\ell) = \bar{d}(k\ell)\bar{S}(\theta-\tau\ell) \tag{4.37}$$

Since the functions $f_0(\theta-\tau\ell)$ of the sum

$$S(\theta) = \sum_{\ell=1}^{k} d_\ell f_0(\theta-\tau\ell)$$

do not overlap (by definition), we can make the following identifications

$$d_\ell = \bar{d}(k\ell)$$

$$f_0(\theta-\tau\ell) = \bar{S}(\theta-\tau\ell)1_\ell(\theta) \tag{4.38}$$

Then, $S(\theta) = \overline{S}(\theta)$ for all $\theta \leq k\tau$. Now we can go back to the general criterion (4.3):

$$J = \lim_{T\to\infty} \frac{1}{T}\int_0^T dt \int_0^t tr[(H(t)\overline{S}(\theta) - \overline{H}(t)\overline{S}(\theta))^T (H(t)\overline{S}(\theta) - \overline{H}(t)\overline{S}(\theta))]d\theta =$$

$$= \lim_{T\to\infty} \frac{1}{T}\int_0^T dt \int_0^t tr[\overline{S}^T(\theta)(H(t)-\overline{H}(t))^T (H(t)-\overline{H}(t))\overline{S}(\theta)]d\theta =$$

$$= \lim_{T\to\infty} \frac{1}{T}\int_0^T dt \int_0^t tr[\overline{S}(\theta)\overline{S}^T(\theta)(H(t)-\overline{H}(t))^T (H(t)-\overline{H}(t))]d\theta =$$

$$= \lim_{T\to\infty} \frac{1}{T}\int_0^T dt\; tr[\sum_{\ell=1}^{k} \int_{\tau(\ell-1)}^{\tau\ell} \overline{S}(\theta)\overline{S}^T(\theta)d\theta \cdot \tag{4.39}$$

$$\cdot (H(t) - \overline{H}(t))^T (H(t) - \overline{H}(t))] + \tilde{J},$$

$$\tilde{J} = \lim_{T\to\infty} \frac{1}{T}\int_0^T dt\; tr[\int_{k\tau}^{t} \overline{S}(\theta)\overline{S}^T(\theta)(H(t)-\overline{H}(t))^T (H(t)-\overline{H}(t))]d\theta$$

$$J-\tilde{J} = \lim_{T\to\infty} \frac{1}{T}\sum_{k=0}^{\infty} \int_{k\tau}^{(k+1)\tau} dt \sum_{\ell=1}^{k} tr[C_\ell (H(t) - H(t))^T \cdot (H(t) - H(t))] \tag{4.40}$$

$C_\ell \triangleq \int_{\tau(\ell-1)}^{\tau\ell} \overline{S}(\theta)\overline{S}^T(\theta)d\theta$ is a KxK matrix. As before, each component of the sum over k depends only on value of $(H(t) - \overline{H}(t))^T(H(t) - \overline{H}(t))$ on interval $k\tau < t \leq (k+1)\tau$. So, we can minimize each term separately.

$$J_k = \int_{k\tau}^{(k+1)\tau} dt \sum_{\ell=1}^{k} tr[C_\ell (H(t)-\overline{H}(t))^T (H(t)-\overline{H}(t))] =$$

$$= tr[\sum_{\ell=1}^{k} C_\ell \int_{k\tau}^{(k+1)\tau} (H(t)-\overline{H}(t))^T (H(t)-\overline{H}(t))dt] =$$

$$= \sum_{\ell=1}^{k} tr[C_\ell \int_{k\tau}^{(k+1)\tau} (H(t)-\overline{H}(t))^T (\overline{H}(t)-H(t)dt] \tag{4.41}$$

$$= \sum_{\ell=1}^{k} J_{k\ell}$$

This implies that quantity

$$J_{k\ell} = tr[C_\ell \int_{k\tau}^{(k+1)\tau} (H(t)-\overline{H}(t))^T (H(t)-\overline{H}(t))dt] \tag{4.42}$$

is to be minimized by choosing $H(t)$.

Suppose we look for $H(t)$ in the form

$$H(t-k\tau) = g_0(t-k\tau)e_k \tag{4.43}$$

where $g_0(t-k\tau)$ and e_k are NxL_1 and L_1xK matrices, respectively. Then,

$$J_{k\ell} = tr[C_\ell \int_{k\tau}^{(k+1)\tau} (e_k^T g^T(t-k\tau)g(t-k\tau)e_k -$$

$$- 2e_k^T g^T(t-k\tau)\overline{H}(t) + \overline{H}^T(t)\overline{H}(t))dt] \tag{4.44}$$

Introduce two matrices

$$T_0 \triangleq \int_{k\tau}^{(k+1)\tau} g_0^T(t-k\ )g_0(t-k\ )dt = \int_0^{\tau} g_0^T(t)g_0(t)dt$$

$$\psi_k \triangleq \int_{k\tau}^{(k+1)\tau} g_0^T(t-kt)\overline{H}(t)dt = \int_0^{\tau} g_0^T(t)H(t+k\ )dt \tag{4.45}$$

Now,

$$J_{k\ell} = \mathrm{tr}[C\ (e_k^T T_0 e_k - 2e_k^T{}_k + {}_{k\tau}^{(k+1)\tau}\ \bar{H}^T(t)\bar{H}(t)dt)]$$

$$\frac{\partial J_{k\ell}}{\partial e_k} = 2T_0 e_k C_\ell - 2\psi_k C_\ell = 0 \tag{4.47}$$

(see formulas IA and IB of Appendix). Since C_ℓ is arbitrary, i.e. independent of hold device characteristics, we conclude

$$e_k = T_0^{-1}\psi_k$$

if $T^{=1}$ exists.

For the specific case where

$$g_0(t) = Ce^{At}$$
$$\bar{H}(t) = C_e^{At} \tag{4.49}$$

we have be definition (4.45)

$$T_0 = \int_0^\tau e^{A^T t}\ C^T C e^{At} dt$$

$$\psi_k = \int_0^\tau e^{A^T t}\ C^T C\ e^{At} e^{Ak\tau} dt = T_0 e^{Ak\tau} \tag{4.50}$$

The matrix T_0 is full rank if (A,C) is observable [1]. Formula (4.48) then yields

$$e_k = e^{Ak\tau} \tag{4.51}$$

This result is expected; it corresponds to the case of exact approximation of hold device and has been derived before using simpler considerations (formula 4.21).

The general case

In general, both the sampling and the hold dervices may not be matched perfectly outside the "triangle strip".

Given a desired structure for these circuits, i.e. functions $f_0(\theta-\tau\ell)$ and $g_0(t-k\ )$, we might then be interested in determining a computational procedure (matrices d_ℓ and e_k) which provides an optimal (in the above sense) hybrid approximation of a nominal system.

So, we have for the sampler and hold, respectively

$$S(\theta) = \sum_{\ell=1}^{k} d_\ell f_0(\theta-\tau\ell)$$

$$H(t-k\tau) = g_0(t-k\tau)e_k \tag{4.52}$$

The criterion (4.3) is

$$J = \lim_{T\to\infty} \frac{1}{T}\int_0^T dt \int_0^t tr[(g_0(t-k\tau)e_k\sum_{\ell=1}^{k} d_\ell f_0(\theta-k\ell) - \overline{H}(t)\overline{S}(\theta))^T \cdot$$

$$\cdot\ g_0(t-k\tau)e_k\sum_{m=1}^{k} d_m f_0(\theta-\tau\ell) - \overline{H}(t)\overline{S}(\theta))]d\theta =$$

$$= \lim_{T\to\infty} \frac{1}{T}\int_0^T dt \int_0^t tr[(\sum_{\ell=1}^{k} f_0^T(\theta-\tau\ell)d_\ell^T e_k^T g_0^T(t-k\tau) -$$

$$-\overline{S}^T(\theta)\overline{H}^T(t))\ \cdot\ g(t-k\tau)e_k\sum_{m=1}^{k} d_m f_0(\theta-\tau m) - \overline{H}(t)\overline{S}(\theta))]d\theta =$$

$$= \lim_{T\to\infty} \frac{1}{T} \int_0^T dt \int_0^t \mathrm{tr}[\sum_{\ell=1}^{k}\sum_{m=1}^{k} f_0^T(\theta-\tau\ell) \cdot d_\ell^T e_k^T g_0^T(t-k\tau) g_0(t-k\tau) e_k$$

$$\cdot\, d_m f_0(\theta-\tau m) - 2\overline{S}^T(\theta)\overline{H}^T(t) g_0(t-k\tau) e_k \sum_{m=1}^{k} d_m f_0(\theta-\tau m) +$$

$$+\, \overline{S}^T(\theta)\overline{H}^T(t)\overline{H}(t)\overline{S}(\theta)]d\theta$$

Consider the sum

$$\sum_{m=1}^{k}\sum_{\ell=1}^{k} \mathrm{tr}[f_0^T(\theta-\tau\ell) d_\ell^T e_k^T g_0^T(t-k\tau) g_0(t-k\tau) \cdot e_k d_m f_0(\theta-\tau m)$$

$$= \sum_{m=1}^{k}\sum_{\ell=1}^{k} \mathrm{tr}[f_0(\theta-\tau m) \cdot f_0^T(\theta-\tau\ell) d_\ell^T e_k^T g_0^T(t-k\tau) g_0(t-k\ell) e_k d_m] \tag{4.54}$$

Due to the definition (Fig. 3) of functions $f_0(\theta)$ we can write

$$f_0(\theta-\tau m) f_0^T(\theta-\tau\ell) = f_0(\theta-\tau\ell) f_0^T(\theta-\tau\ell)\, \delta_{m\ell} \tag{4.55}$$

Then (4.54) takes the form

$$\sum_{\ell=1}^{k} \mathrm{tr}[f_0(\theta-\tau\ell) f_0^T(\theta-\tau\ell) d_\ell^T e_k^T g_0^T(t-k\tau) g_0(t-k\tau) e_k d_\ell]$$

and for J we have

$$J = \lim_{N\to\infty} \frac{1}{N} \sum_{k=0}^{N} \sum_{\ell=1}^{k} \mathrm{tr}[F_0 d_\ell^T e_k^T T_0 e_k d_\ell -$$

$$- 2\phi_\ell d_\ell^T e_k^T \psi_k + \int_0^\infty dt \int_0^{kT} \sum_{\ell=1}^{k} \bar{S}^T(\theta)\bar{H}^T(t) \cdot \bar{H}(t)\bar{S}(\theta)d\theta] + \tilde{\tilde{J}} \tag{4.56}$$

where as before, $\tilde{J}$ is

$$\tilde{J} = \lim_{T\to\infty} \frac{1}{T} \int_0^T dt \int_{kT}^{t} tr[(H(t)S(\theta) - \bar{H}(t)\bar{S}(\theta))^T (H(t)S(\theta) - \bar{H}(t)\bar{S}(\theta))]d\theta \tag{4.57}$$

The quantity $J - \tilde{J}$ is to be minimized by choosing matrices e_k and d_ℓ. We can now introduce a new matrix of computer coefficients

$$D_{k\ell} = e_k d_\ell \tag{4.58}$$

Its dimension is $L_1 x L$. With this definition, $J - \tilde{J}$ again breaks down into independent optimization squares with costs

$$J_{k\ell} = tr[F_0 D_{k\ell}^T T_0 D_{k\ell} - 2\psi_k D_{k\ell}\phi_\ell] \tag{4.59}$$

$$\frac{\partial J_{k\ell}}{\partial D_{k\ell}} = 2T_0 D_k\ F_0 - 2\psi_k \phi_\ell = 0 \tag{4.60}$$

and

$$D_{k\ell} = T_0^{-1}\psi_k \phi_\ell F_0^{-1} \tag{4.61}$$

The overall optimal hybrid operator now is

$$G(t,\theta) = g_0(t-kT)\sum_{\ell=1}^{k} D_{k\ell} f_0(\theta-T\ell) \tag{4.62}$$

where $g_0(t-k\ )$ and $f_0(\theta-T\ell)$ represent simplified sampling and hold structures. The simplification means either lower dimension of analog circuits or simplified feedback loops or both.

The optimal hybrid system is shown in Fig. 17.

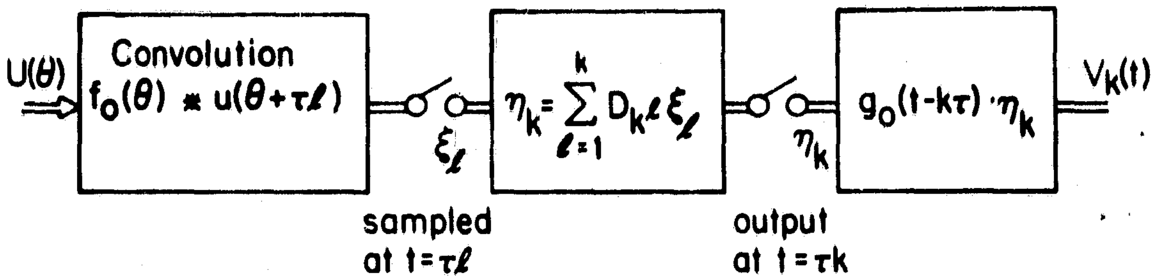

Fig. 17. Optimal Hybrid Approximation.
General Case.

<u>Example</u>.

Single input-single output system. Suppose,

$$\bar{G}(t,\theta) = Ce^{A(t-\theta)}f \tag{4.63}$$

where A is NxN matrix, f and C^T and N-vectors. Let $f_0(\lambda)$ and $g_0(\lambda)$ be chosen in the form

$$f_0(\lambda) = \begin{cases} x(\lambda)\cdot I\ , & -\tau \leq \lambda < 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.64}$$

$$g_0(\lambda) = \begin{cases} 1 & 0 \leq \lambda < \tau \\ 0 & \text{otherwise} \end{cases}$$

where $x(\theta)$ is a scalar function.

Then,

$$\overline{H}(t) = Ce^{At} \quad ; \qquad \overline{S}(\theta) = e^{-\Lambda\theta} \cdot f$$

$$F_0 = \int_{-\tau}^{0} x^2(\theta)d\theta \cdot I$$

$$T_0 = \tau$$

$$\phi_\ell = e^{-\Lambda\tau\ell} \int_0^{\tau} e^{-\Lambda\theta} f \; x(\theta)d\theta$$

$$\psi_k = C \int_0^{\tau} e^{At}dt \cdot e^{\Lambda\tau k} = -CA^{-1}(I - e^{\Lambda\tau})e^{\Lambda\tau}$$

$$D_{k\ell} = T^{-1}\psi_k \, \phi_\ell F^{-1} =$$

$$= \frac{1}{\tau} \frac{1}{\int_0^{\tau} x^2(\theta)d\theta} CA^{-1}(I - e^{-A\tau}) \cdot e^{-A\tau(k-\ell)} \bullet \int_0^{\tau} e^{\Lambda\theta} x(\theta)d \cdot f \qquad (4.66)$$

The expression (4.66) of optimal scalar sequence of coefficients is to be used in the digital computer of Fig. 17.

SECTION 5

Robustness of Hybrid Systems

In Section 4 we have shown how continuous time linear operators may be approximated by a hybrid system. If this approximation takes place in a feedback loop of a closed loop control system, the stability and performance properties will degrade somewhat due to errors of approximation.

The extent to which such errors effect stability has been investigated in general by Safonov [3]. Sufficient stability conditions for continuous and sampled data systems have been established. In reference [4], simpler frequency domain conditions have been derived for linear time invariant systems subject to additive errors of various kinds. A block diagram of these systems is shown on Fig. 16. $\bar{L}$ is a nominal time invariant operator, and ΔL is a perturbation or error, time invariant as well.

Then, according to [4], the system remains stable if

$$\underline{\sigma}(I+\bar{L}(j\omega)) > \bar{\sigma}(\Delta L(j\omega)) \tag{5.1}$$

Here $L(J\omega)$ and $\Delta L(j\omega)$ are transfer functions of $\bar{L}$ and ΔL, and $\bar{\sigma}(A)$, $\underline{\sigma}(A)$ denote maximum and minimum singular values of A.

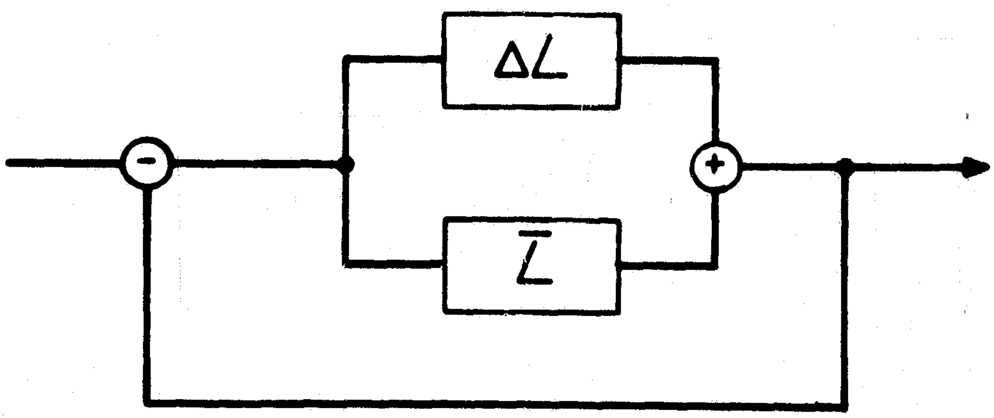

Fig. 18. Feedback Loop with Perturbations.

Figure 18 may be viewed as a general diagram for the specific control system shown in Fig. 19.

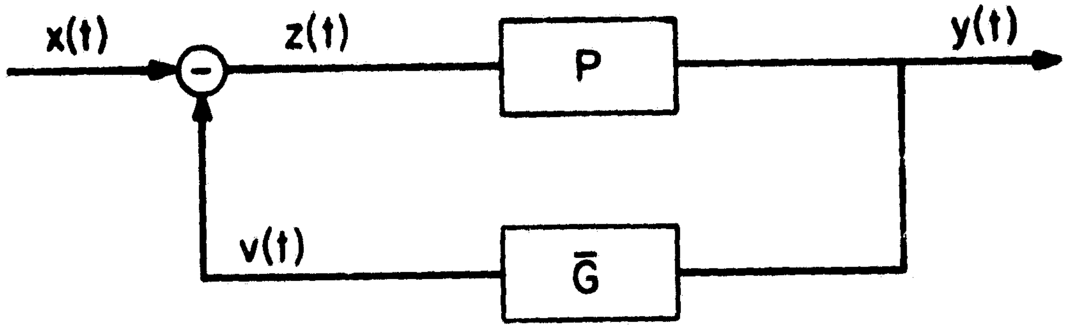

Fig. 19. Control System

Here P and $\bar{G}$ are time invariant continuous matrix convolution operators. That is

$$y(t) = (Pz)(t) = \int_0^t P(t-\lambda)z(\lambda)d\lambda \tag{5.2}$$

and

$$v(t) = \int_0^t \bar{G}(t-\lambda)y(\lambda)d\lambda$$

P usually represents the "plant" to be controlled, and $\bar{G}$ is the ideal continuous time controller designed to regulate P. This system can also be drawn as in Fig. 20.

It has overall operator $P\bar{G}$:

$$(P\bar{G}z)(t) = \int_0^t P\bar{G}(t-\theta)z(\theta)d\theta \tag{5.3}$$

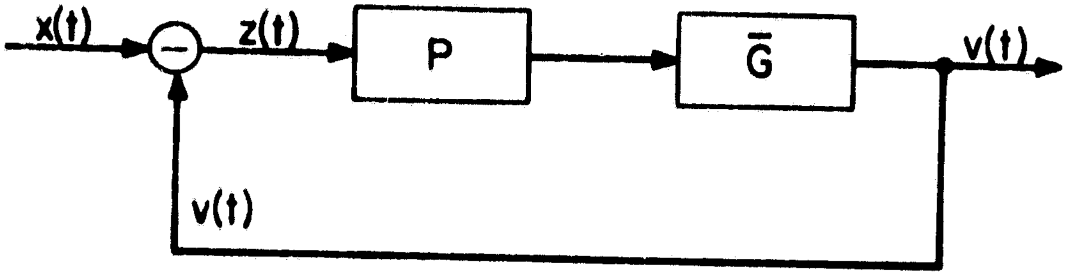

Fig. 20. Control System

$$P\overline{G}(t) = \int_0^t \overline{G}(t-\theta)P(\theta)d\theta \tag{5.4}$$

is again a time invariant matrix convolution operator. Our objective in this section will be to examine the effect on stability caused by approximating the operator $\overline{G}$ with a hybrid system, i.e. what happens if we use a digitally implemented controller? The tool for this analysis will be the stability robustness condition (5.1). Unfortunately, the condition (5.1) cannot be applied directly to the approximated system because a hybrid system is inherently time-variant in nature. This follows, for example, from errors in the "triangle strip", as discussed in Section 2.

However, if we construct a time-invariant "bounding operator" W with property

$$||w_u|| \geq ||\Delta Gu|| \qquad \forall\ u \tag{5.5}$$

where $\Delta G = \overline{G}-G$, then stability condition (5.1) may be applied in a form

$$\underline{\sigma}(I + P\overline{G}(j\omega)) > \overline{\sigma}(W(j\omega))\overline{\sigma}(P(j\omega)) \tag{5.6}$$

for all $\omega > 0$, where $W(j\omega)$ denotes a matrix of Fourier transfc ͜ of $W(\theta)$.

We note from Fig. 10 that an optimal unconstrained hybrid approximation of a linear time-invariant operator satisfies

$$G(t,\theta) = \overline{G}(t,\theta) \qquad \text{for all } t > 0 \text{ and } \theta \leq \theta < k\tau$$

and

$$G(t,\theta) = 0 \qquad \text{for } k\tau < \theta \leq t \tag{5.7}$$

Therefore,

$$\Delta G = \begin{cases} 0 & \text{if} \quad 0 < \theta < k\tau \\ \overline{G}(t,\theta) & \text{if} \quad k\tau < \theta \leq t \end{cases} \tag{5.8}$$

i.e. $\Delta G \neq 0$ only inside the"triangle strip".

The approach may be suggested to construct the operator W for general multivariable systems. This approach provides a rather conservative upper bound for $||\Delta G||$.

For $(\Delta Gu)(t)$ we have

$$(\Delta Gu)(t) = \int_{k\tau}^{t} \overline{G}(t-\lambda)u(\lambda)d\lambda \tag{5.9}$$

$$\begin{aligned} ||(\Delta Gu)(t)|| &\leq \int_{k\tau}^{t} ||\overline{G}(t-\lambda)u(\lambda)||d\lambda \leq \\ &\leq \int_{k\tau}^{t} \overline{\sigma}[\overline{G}(t-\lambda)]\,||u(\lambda)||d\lambda \\ &\leq C \int_{k\tau}^{t} ||u(\lambda)||d\lambda \end{aligned} \tag{5.10}$$

where $C \triangleq \max\limits_{0<\theta<\tau} \overline{\sigma}[\overline{G}(\theta)]$,

$\bar{\sigma}(A)$ is a maximum singular value of A. Then,

$$||\Delta Gu||^2 = \int_0^\infty ||(\Delta Gu)(t)||^2 dt =$$

$$= \sum_{k=0}^{\infty} \int_{k\tau}^{(k+1)\tau} ||(\Delta Gu)(t)||^2 dt \leq$$

$$\leq \sum_{k=0}^{\infty} \int_{k\tau}^{(k+1)\tau} c^2 \left(\int_{k\tau}^{t} ||u(\lambda)|| d\lambda \right)^2 dt \leq \tag{5.11}$$

$$\leq \sum_{k=0}^{\infty} c^2 \tau \left(\int_{k\tau}^{(k+1)\tau} u(\lambda) d\lambda \right)^2 \leq$$

$$\leq c^2 \tau^2 ||u||^2$$

Hence,

$$||\Delta Gu|| \leq c\tau ||u|| \tag{5.12}$$

This approach provides general but conservative stability condition

$$\underline{\sigma}(I + P\overline{G}(j\omega)) \geq \tau c\, \overline{\sigma}(P(j\omega)) \tag{5.13}$$

where constant C is defined by (5.12). This means that the curve $\sigma(I+P\overline{G}(j\omega))$ is to be compared with a certain level $\tau c \overline{\sigma} P(j\omega)$, as long as it is strictly above that level the system's stability with G replacing $\overline{G}$ is assured.

In the next section we shall explore a particular three-dimensional single input-single output control system and this approach to the robustness problem will be applied.

SECTION 6

A Hybrid Control System Example

In this section we consider a three dimensional single input - single output control system

$$\dot{x}(t) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} x(t) + \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} u(t) \tag{6.1}$$

with feedback control law

$$u(t) = -(\omega_0^3 \quad 2\omega_0^2 \quad 2\omega_0)x(t) \tag{6.2}$$

with $\omega_0 = 1\ \mathrm{sec}^{-1}$. The system may be drawn as shown in Fig. 2.1

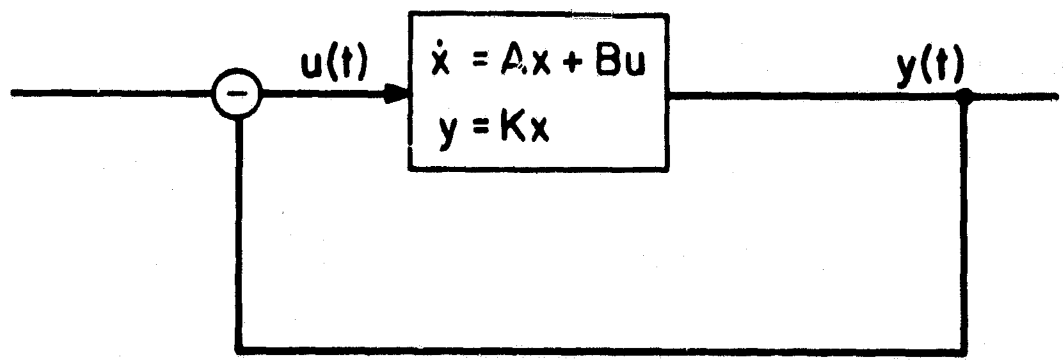

Fig. 21. Control System Example

where A and B are given in (6.1) and

$$K = -(1 \quad 2 \quad 2) \tag{6.3}$$

The control law (6.2) for system (6.1) minimizes a cost function

$$\int_0^\infty (x^T Q x + u^2)\,dt \tag{6.4}$$

where

$$Q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Let this system represent the nominal design operator $\overline{G}$ discussed in previous sections with $P = 1$. Then

$$\overline{G}(\theta) = (1 \quad 2 \quad 2)\, e^{\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}\theta} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} =$$

$$= Ke^{A\theta}B$$

For $e^{A\theta}$ we can get a closed form expression (since $A^3 = 0$):

$$e^{A\theta} = \begin{pmatrix} 1 & \theta & \frac{\theta^2}{2} \\ 0 & 1 & \theta \\ 0 & 0 & 1 \end{pmatrix} \tag{6.6}$$

Therefore

$$\overline{G}(\theta) = (1 \quad 2 \quad 2) \begin{pmatrix} 1 & \theta & \frac{\theta^2}{2} \\ 1 & 0 & \theta \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \tag{6.7}$$

$$= \frac{\theta^2}{2} + 2\theta + 2$$

For the left hand part of the stability inequality (5.1) we have

$$G(j\omega) = K(Ij\omega - A)^{-1}B =$$

$$= (1 \quad 2 \quad 2) \begin{pmatrix} j\omega & -1 & 0 \\ 0 & j\omega & -1 \\ 0 & 0 & j\omega \end{pmatrix}^{-1} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{6.8}$$

So,

$$\bar{G}(j\omega) = (1 + 2j\omega - 2\omega^2)\frac{j}{\omega^3} \tag{6.10}$$

and

$$\underline{\sigma}(1 + \bar{G}(j\omega)) = |1 + \bar{G}(j\omega)| \ ;$$

$$|1 + G(j\omega)|^2 = \left(1 - \frac{2}{\omega^2}\right)^2 + \left(\frac{1}{\omega^3} - \frac{2}{\omega}\right)^2 \tag{6.11}$$

The Fig. 22 shows the $\underline{\sigma}(\omega)$

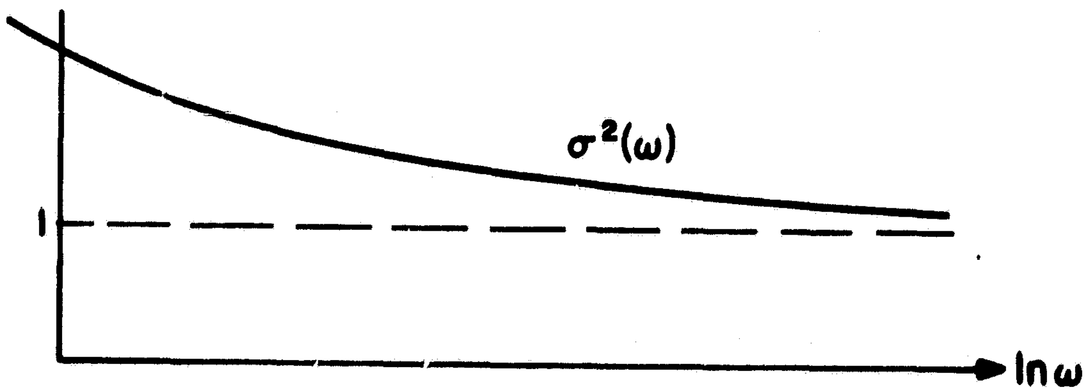

Fig. 22. Dependence $\underline{\sigma}(\omega)$.

Since $P = 1$, the right hand part of the criterion (5.1) for this example is independent of ω. We can rewrite the condition (5.1) in a form

$$\min_{\omega} |1 + \bar{G}(j\omega)| > \bar{\sigma}$$

with, as can be seen from the expression (6.11),

$$\min_{\omega} |1 + \bar{G}(j\omega)| = 1 \tag{6.13}$$

According to (5.13), therefore, we have

$$1 > \bar{\sigma} = \tau \max_{0<\theta\leq\tau} \bar{G}(\theta) =$$

$$= \tau \max_{0<\theta\leq\tau}\left(\frac{\theta^2}{2} + 2\theta \; R \; 2\right) = \tau\left(\frac{\tau^2}{2} + 2\tau + 2\right) \tag{6.14}$$

So, the stability condition is

$$1 > \tau\left(\frac{\tau^2}{2} + 2\tau + 2\right) \tag{6.15}$$

The critical value for τ then is

$$\tau_{cr}\left(\frac{\tau_{cr}^2}{2} + 2\tau_{cr} + 2\right) = 1 \tag{6.16}$$

$$\tau_{cr} = 0.36 \text{ sec}$$

This number should be interpreted as a sample time τ_{cr} which is sufficiently small to guarantee stability of the control system when the continuous operator $\bar{G}$ is replaced by an optimal hybrid approximation G.

An implementation of the resulting closed loop control system is shown in Fig. 23.

It is clear from this figure that

$$y(t) = Ke^{A(t-k\tau)}\eta_k \tag{6.17}$$

where

$$\eta_k = e^{A\ k}\sum_{\ell=1}^{k} e^{-A\ell\tau}\xi_\ell$$

$$= \sum_{\ell=1}^{k} e^{A\tau(k-\ell)}\xi_\ell \tag{6.18}$$

and

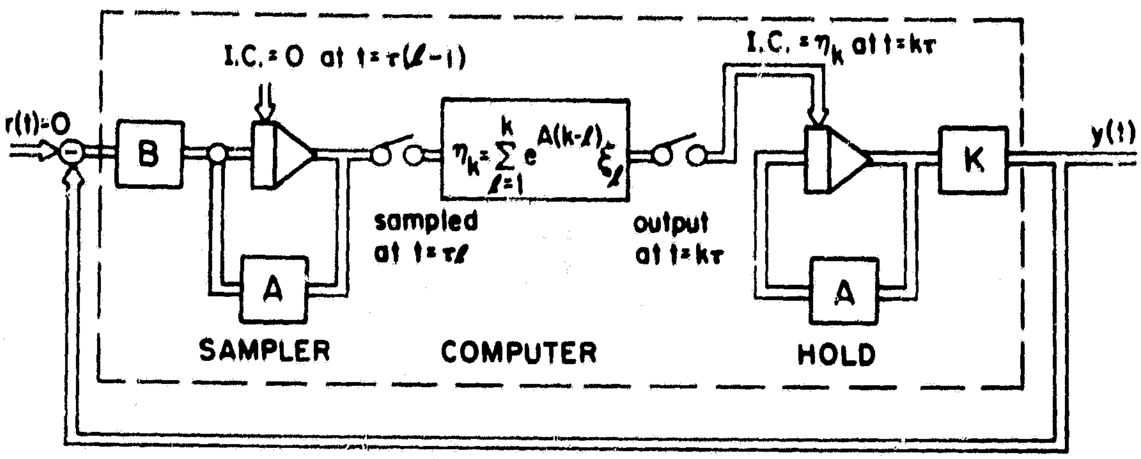

Fig. 23. Feedback Control System with Hybrid Operator

$$\xi_\ell = \int_{(\ell-1)\tau}^{\ell\tau} e^{A(\tau\ell-\theta)} Bu(\theta)d\theta \tag{6.19}$$

Hence we can rewrite (6.17) as

$$\begin{aligned} y(t) &= Ke^{A(t-t_k)} \int_0^{k\tau} e^{(k\tau-\lambda)} Bu(\lambda)d\lambda = \\ &= K \int_0^{k\tau} e^{A(t-\lambda)} Bu(\lambda)d\lambda \end{aligned} \tag{6.20}$$

The only difference between this expression and the continuous operator $\bar{G}$ is in integration limit. This indicates that G and $\bar{G}$ coincide exactly outside the "triangle strip" within which they hybrid operator equals to zero. This property has been discussed in Section 4.

A discrete time equation for the closed loop behavior of the approximated system in Fig. 23 can be written as follows:

$$\eta_k = e^{A\tau}\eta_{k-1} + \xi_k$$

$$\xi_k = \int_{\tau(k-1)}^{\tau k} e^{A(\tau k-\theta)} By(\theta)\, d\theta \qquad (6.21)$$

$$y(\theta) = Ke^{A(\theta-(k-1)\tau)}\eta_{k-1}$$

Hence

$$\eta_k = \left[e^{A\tau} - \int_{\tau(k-1)}^{\tau k} e^{A(\tau k-\theta)} BKe^{A(\theta-(k-1)\tau)} d\theta\right] \cdot \eta_{k-1} \qquad (6.22)$$

Therefore, the evolution of η_k corresponds to discrete system with transition matrix

$$\phi_1(\tau) = e^{A\tau} - \int_0^{\tau} e^{A(t-\lambda)} BKe^{A\lambda} d\lambda \qquad (6.23)$$

This matrix can be evaluated analytically in order to obtain its characteristic equation

$$\phi_1(\tau) = \begin{pmatrix} 1 - \frac{\tau^3}{6} & \tau - \frac{\tau^3}{3} - \frac{\tau^4}{24} & \frac{\tau^2}{2} - \frac{\tau^3}{3} - \frac{\tau^4}{12} \\ -\frac{\tau^2}{2} & 1 - \tau^2 - \frac{\tau^3}{6} & \tau - \tau^2 - \frac{\tau^3}{3} - \frac{\tau^4}{24} \\ -\tau & -2\tau - \frac{\tau^2}{2} & 1 - 2\tau - \tau^2 - \frac{\tau^3}{3} \end{pmatrix} \qquad (6.24)$$

Eigenvalues of the matrix (6.24) have been computed for different τ. Its values indicate that the system looses its stability when $\tau = 0.54$ sec. This compares quite favorably with the sufficient stability sound.

From the viewpoint of closed loop stability, the hybrid approximation $\bar{G} \approx G$ is not necessarily a good one. This is evident when we examine an alternate hybrid approximation which duplicates closed loop rather than open loop behavior outside the "triangle strip". This can be done by changing the hold device as follows:

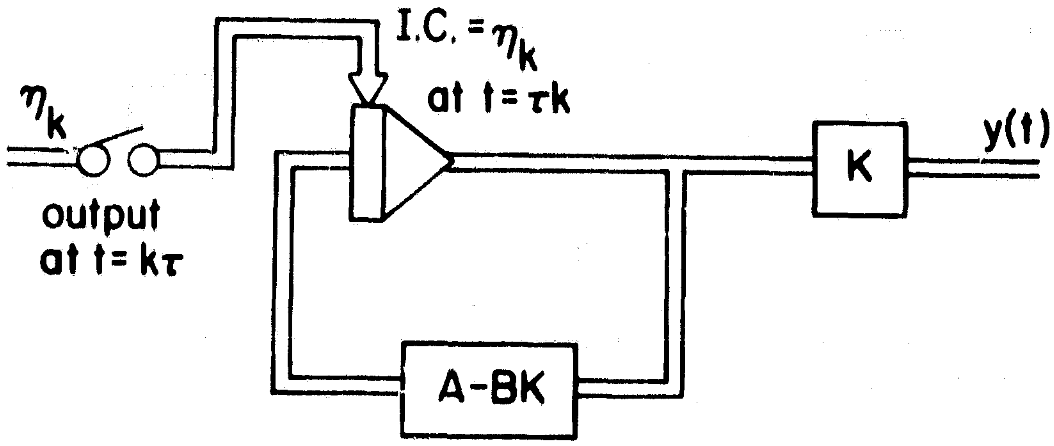

Fig. 24. Hold Device To Approximate Closed Loop System.

Note that this "hold" generates exactly the same output signal y(t) as the continuous closed loop system under deterministic assumptions. This output is

$$y(t) = Ke^{(A-BK)(t-k\tau)}x(k\tau) \tag{6.25}$$

In order to find a transition matrix of the corresponding discrete system, we write

$$\eta_k = e^{A\tau}\eta_{k-1} + \xi_k$$

$$\xi_k = \int_{\tau(k-1)}^{\tau k} e^{A(\tau k-\lambda)} BKe^{(A-BK)\lambda} \eta_{k-1} d\lambda \tag{6.26}$$

$$\eta_k = \phi_2(\tau)\eta_{k-1}$$

This leads to the transition matrix

$$\phi_2(\tau) = e^{A\tau} - \int_0^{\tau} e^{A(\tau-\lambda)} BKe^{(A-BK)\lambda} d\lambda \tag{6.27}$$

Using the identity IY A in the Appendix, (6.27) may be simplified to

$$\phi_2(\tau) = e^{(A-BK)\tau} \tag{6.28}$$

This result is natural because the system (6.25) was constructed to exhibit exactly the same closed loop behavior as the continuous one under deterministic assumptions. Therefore, this system must remain stable for all τ.

In order to compare performances of systems (6.17) and (6.25) in noisy situations, both approximations were examined for white noise inputs $\nu(t)$ with intensity matrix

$$I_\nu = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{6.29}$$

inserted at point ξ_ℓ in Fig. 23.

Convariance matrices of discrete systems (6.22) and (6.26) propogate according to the Lyapunov equation

$$\begin{aligned} P_{k+1}^{(1)} &= \phi_1(\tau)P_k^{(1)}\phi_1(\tau) + \overline{I}_\nu(\tau) \\ P_{k+1}^{(2)} &= \phi_2(\tau)P_k^{(2)}\phi_2(\tau) + \overline{I}_\nu(\tau) \end{aligned} \tag{6.30}$$

where $\overline{I}_\nu(\tau)$ is intensity matrix of noise accumulated on the sample interval τ.

$$\overline{I}_\nu = \int_0^\tau e^{A\lambda} I_\nu e^{A^T\lambda} d\lambda =$$

$$= \int_0^\tau \begin{pmatrix} 1 & \lambda & \frac{\lambda^2}{2} \\ 0 & 1 & \lambda \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ \lambda & 1 & 0 \\ \frac{\lambda^2}{2} & \lambda & 1 \end{pmatrix} d\lambda$$

$$\begin{pmatrix} \frac{\tau^5}{20} & \frac{\tau^4}{8} & \frac{\tau^3}{6} \\ \frac{\tau^4}{8} & \frac{\tau^3}{3} & \frac{\tau^2}{2} \\ \frac{\tau^3}{6} & \frac{\tau^2}{2} & \tau \end{pmatrix} \tag{6.31}$$

and where $P_k^{(i)}$ denote the state covariance of the two systems

$$P_k^{(i)} = E[\eta_k^{(i)} \eta_k^{(i)T}], \quad i = 1,2 \tag{6.32}$$

Then, assuming steady state behavior of both systems we can solve matrix equation (6.30) to obtain $P^{(1)}$ and $P^{(2)}$.

$$P^{(1)} = \phi_1(\tau) P^{(1)} \phi_1^T(\tau) + \overline{I}_\nu(\tau)$$

$$P^{(2)} = \phi_2(\tau)\ P^{(2)} \phi_2^T(\tau) + \overline{I}_\nu(\tau) \tag{6.33}$$

We used diagonal terms of the matrices $P^{(i)}$ to compare systems performance. The results indicate that the first system (6.22) has a better performance (smaller variance of state variables). It is shown in Fig. 25, for values of $P^{(1)}_{jj}$ and $P^{(2)}_{j1}$.

P_{11}		P_{22}		P_{33}		
I	II	I	II	I	II	
0.2251	0.3292	$7{:}124.10^{-2}$	$9.101.10^{-2}$	0.3025	0.3327	T=0.2 sec
0.3237	0.8633	$5.062.10^{-2}$	$3.425.10^{-2}$	0.3759	0.4579	T=0.4 sec

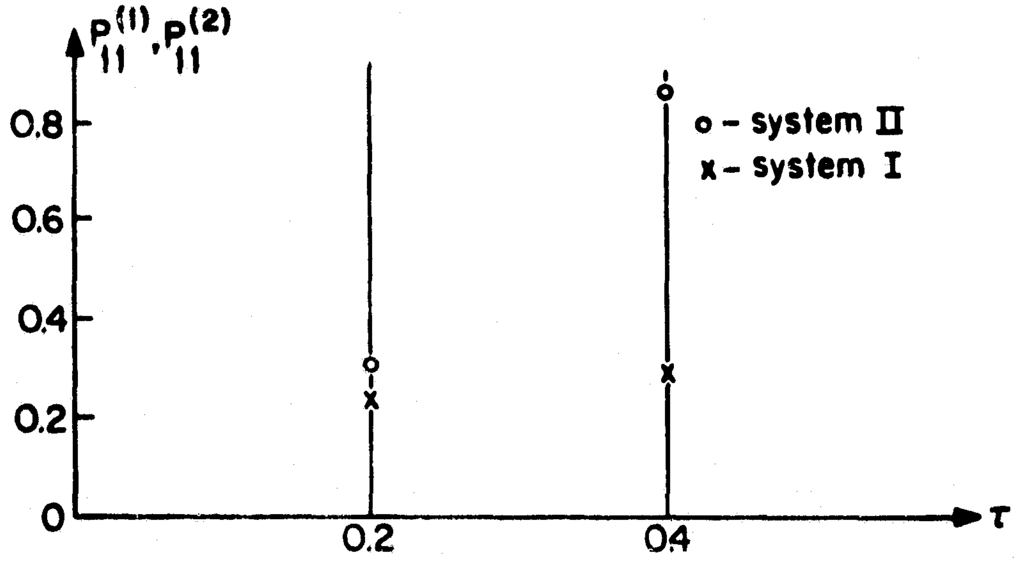

Fig. 25. Comparison of Two Discrete Systems Performance

This result may be interpreted in the following way. As we know from Section 4 the approximation used for the first system minimizes the cost

criterion (4.1) which minimizes mean square deviation between the outputs of the nominal and the hybrid system when they subjected to the same white noise process. Therefore, all other hybrid systems yield a greater value of cost function against this criterion.

LIST OF USED SYMBOLS AND THEIR DIMENSIONS

τ, t, θ	scalars
k, ℓ	integer variables
$G(t,\theta)$, $G(t,\theta)$ - NxM	matrices
$\overline{H}(t)$ - NxK	matrix
$H(t)$ - NxK	matrix
$\overline{S}(\theta)$, $S(\theta)$ - KxM	matrices
$M(t)$, M_k - KxK	matrices
$V(t)$, $V_k(t)$ - N	vectors
$u(\theta)$ - M	vector
$f(\theta)$ - LxN	matrix
ξ_ℓ - L	vector
d_ℓ - KxL	matrix
e_k - L_1xK	matrix
$g(t)$ - NxL_1	matrix
F - LxL	matrix
ϕ_ℓ - KxL	matrix
η_k - L_1	vector
T - L_1xL_1	matrix
ψ_k - L_1xK	matrix
$D_{k\ell}$ - L_1xL	matrix
B_k - KxK	matrix
C_ℓ - KxK	matrix

$\alpha, \varepsilon, \Delta_k, A_1, A_2, r$ - scalar parameters

$\alpha, \varepsilon, \Delta_k, \Lambda_1, \Lambda_2, r$ - scalar parameters

$\gamma(\theta)$ - scalar function

Λ - NxN matrix

ρ - LxM matrix

β - M vector

Conclusions

The general representation of the hybrid operator in a continuous time form is obtained in the first section of the thesis.

Both lower and upper for the hybrid operator are established and their dependence on sampling interval ε is clarified. Different types of sampling are discussed.

A general approach for hybrid approximation of a continuous time invariant multivariable systems is suggested with particular type of minimization criterion. This criterion is physically interpreted. Also some examples are considered in details. The formula for optimal computer coefficients is derived to provide a minimum value of criterion given fixed sampler and hold structures. This may lead to simplified analog circuits. Schemes for implementation are drawn.

A robustness problem for hybrid systems is formulated and investigated. A robustness condition for continuous systems is applied to the hybrid operator and the critical sampling interval value is found. This value guarantees the robustness of the hybrid approximating system.

All these concepts and methods are illustrated on the three-dimensional control system. A numerical simulation has been performed and results interpreted. They indicate a consistency of the found critical value of sampling interval with its exact value.

Directions for Further Investigations

1. In order to get a better understanding of processes in hybrid systems one would investigate how does the fixation of functions $f_0(\lambda)$ and $g_0(\lambda)$ affect the quality of hybrid approximation and the robustness of the approximated system.

2. As mentioned in Section 4, the bounding operator for provides a rather conservative sufficient stability condition and, consequently, relatively small value of τ_{cr}. A better, less restrictive time invariant operator could be possibly found in order to obtain more precise value for τ_{cr}.

APPENDIX

This Appendix brings together some mathematical formulas and their derivations which are used in various sections of the thesis. The include identities for matrix traces and their derivatives, trace inequalities, inequalities for integrals and sums, one matrix identity used in the example.

I. Differentiation of Traces.

Identity:

$$\frac{\partial}{\partial E}\,\mathrm{tr}[CE^TTE] = 2TEC \tag{A}$$

where C and T are square symmetric matrices.

Proof:

$$\mathrm{tr}(CE^TTE] = \sum_i \sum_j (CE^T)_{ij}\;(TE)_{ji} =$$

$$= \sum_i \sum_j \sum_k \sum_m \; C_{ik}E_{jk}T_{jm}E_{mi}$$

$$\frac{\partial}{\partial E_{pq}}\,\mathrm{tr}[CE^TTE] = \sum_i \sum_j \sum_k \sum_m C_{ik}T_{jm}(E_{jk}\delta_{mp}\delta_{iq} +$$

$$+ E_{mi}\delta_{jp}\delta_{kq}) = \sum_i \sum_j \Big(\sum_k C_{ik}T_{jp}E_{jk}\delta_{iq} +$$

$$+\sum_m C_{iq}T_{jm}E_{mi}\delta_{jp}$$

$$= \sum_j T_{jp}\sum_k C_{qk}E_{jk} + \sum_i C_{iq}\sum_m T_{pm}E_{mi} =$$

$$=\sum_j T_{ip}(CE^T)_{qj} + \sum_i C_{iq}(TE)_{pi} =$$

$$= (CE^T T)_{qp} + (TEC)_{pq} = (ETEC)_{pq}$$

Corollary:

$$\frac{\partial}{\partial E}\, tr[ETE^T] = 2ET$$

Proof: put C=I and transpose both parts of the idenity (A):

$$\frac{\partial}{\partial E^T}\, tr[E^T TE] = 2E^T T$$

Call now $E^T = E'$,

$$\frac{\partial}{\partial E'}\, tr[E'TE'^T] = 2E'T$$

which proves (b).

Identity:

$$\frac{\partial}{\partial E}\, tr[CE^T\psi] = \psi C \tag{C}$$

Proof:

$$tr[CE^T\psi] = \sum_i \sum_j C_{ij} \sum_k E_{kj}\psi_{ki} =$$

$$= \sum_i \sum_j \sum_k C_{ij}\psi_{ki}E_{kj}$$

$$\frac{\partial}{\partial E_{pq}} \mathrm{tr}[CE^T\psi] = \sum_i \sum_j \sum_k c_{ij}\psi_{ki}\delta_{kp}\delta_{iq} =$$

$$= \sum_i c_{iq}\psi_{pi} = (\psi C)_{pq}$$

II. Properties of traces.

A) $\mathrm{tr}(A+B) = \mathrm{tr}(A) + \mathrm{tr}(B)$

B) $\mathrm{tr}[\int A(t)dt] = \int \mathrm{tr}[A(t)]dt$

C) $\mathrm{tr}(A) = \mathrm{tr}(A^T)$

D) $\mathrm{tr}(A \cdot B) \leq \mathrm{tr}(A) \cdot \mathrm{tr}(B)$

E) $\mathrm{tr}(A \cdot B) = \mathrm{tr}(B \cdot A)$

Here A and B are square matrices.

III. Inequalities [6]

$$\left| \sum_{i=1}^{n} \alpha_i \right|^2 \leq n \sum_{i=1}^{n} \alpha_i^2 \qquad \text{(A)}$$

$$\left(\int_a^f f(s)dx \right)^2 \leq (f-a) \int_a^f f^2(x)dx \qquad \text{(B)}$$

IV. Matrix Identity

$$\phi(\tau) = e^{A\tau} - \int_0^\tau e^{A(\tau-\lambda)} BKe^{(A-BK)\lambda} d\lambda = e^{(A-BK)\tau} \qquad \text{(A)}$$

Proof: differentiate both parts with respect to τ:

a) left hand part

$$Ae^{A\tau} - \int_0^{\tau} Ae^{A(\tau-\lambda)}BKe^{(A-BK)\lambda}d\lambda -$$

$$- BKe^{(A-BK)\tau} = A\phi(\tau) - BKe^{(A-BK)\tau}$$

b) right hand part

$$(A-BK)e^{(A-BK)\tau}$$

If we now suppose

$$\phi(\tau) = e^{(A-BK)\tau}$$

then we come to the true equality I=I. So, the formula (A) is proved.

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